Counting and combinatorial proofs

Examples of counting things.

- **Simple Addition.** If there are $a$ varieties of soup and $b$ varieties of salad, then there are $a + b$ possible ways to order a meal of soup or salad (but not both soup and salad).
- **Simple Multiplication.** If there are $a$ varieties of soup and $b$ varieties of salad, then there are $ab$ possible ways to order a meal of soup and salad.
- Let $A$ and $B$ be finite sets that are disjoint ($A \cap B = \emptyset$). Then the simple addition principle is equivalent to
  \[ |A \cap B| = |A| + |B|. \]
- Let $A$ and $B$ be finite sets (not necessarily disjoint). Then the simple multiplication principle is equivalent to
  \[ |A \times B| = |A| \cdot |B|. \]
- **Permutations.** A permutation of a collection of objects is a reordering of them. For example, there are six different permutations of the letters $ABC$, namely $ABC, ACB, BAC, CAB, CBA$. In general, if you have $n$ objects, there are $n!$ permutations of the $n$ objects.
- **Permutations taken $r$ at a time.** The number of different three-letter permutations of “LEWISBURG” is $9 \cdot 8 \cdot 7$: 9 choice for the first letter, 8 for the next and 7 for the third. In general you get
  \[ n \cdot (n-1) \cdots (n-r+1) \left( = \frac{n!}{(n-r)!} \right) =: P(n, r). \]
- What about permuting all the letters of “UNION”? Do it carefully by hand and see that it’s $5!/2$: consider the $5!$ permutations of $UNION$. Since I don’t distinguish $N$ and $\hat{N}$, I have to divide by 2. In general we have:

  We are given a collection of balls that are indistinguishable except for color. If there are $a_i$ balls of color $i$, for $i = 1, 2, \ldots, n$, then the number of different ways that these balls can be arranged in a row is
  \[ \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!a_2!\cdots a_n!}. \]

  **The role of division** In the previous formula and examples, there’s some division. Here’s what that means:

  To count the number of ways a joint event occurs, multiply together the number of choices for each sub-event. To rectify uniform overcounting, divide by the overcounting factor.

- **Combinations of $n$ things taken $r$ at a time.** Here is a seemingly different problem: We are ordering a pizza, and there are eight different toppings available (anchovies, garlic, pineapple, sausage, pepperoni, mushroom, olive, and green pepper). We would like to know how many different pizzas can be ordered with exactly three toppings. In contrast to permutations, the order that we choose the toppings does not matter. For example, a "sausage-mushroom-garlic" pizza is indistinguishable from a "mushroom-garlic-sausage" pizza.
To handle this difficulty, we proceed as we did above. If the order did matter, then the number of different pizzas would be the simple permutation $P(8, 3)$, but then we are uniformly overcounting by a factor of $3!$. So the correct answer is 15. This is denoted $\binom{8}{3}$.

- **Examples**
  - $7 \times 6$
  - $12 + 6 + 5$
  - $7^5$
  - $\binom{10}{4}$
  - $P(10, 4)$
  - $5 \cdot \binom{13}{5}$
  - $\binom{17}{8} \cdot \binom{10}{2}$
  - $\binom{17}{4} + \binom{17}{2}$
  - $\binom{17}{10} = \binom{17}{7}$

**Combinatorial proofs.** Claim

$$\binom{n}{r} + \binom{n}{r + 1} = \binom{n + 1}{r + 1}.$$ 

Let’s count the number of committee of size $r + 1$ you can from a group of size $n + 1$. One the one hand, that’s what the RHS counts. If I can show that the LHS counts the same thing, we’d get the equality. Let Nathan be one of the members of the committee. Let’s count the number of committees Nathan is on. There are $n$ folks left and to make a committee of size $r$ (for a total committee size of $r + 1$), I have to choose $\binom{n}{r}$. If Nathan is not on the committee there are $\binom{n}{r + 1}$ options. This is because there are $n$ possible committee members and this time we have to choose $r + 1$ of them.

**Theorem (Binomial Theorem).** For all integers $n$

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$ 

**Combinatorial Proof.** We think about how multiplication takes place in order to understand why the coefficients are what they are. Consider the expansion of, say,

$$(x + y)^7 = (x + y) \cdots (x + y).$$

What’s the coefficient of $x^3y^4$? Out of the 7 binomials, I choose 3 of them. In how many ways can I do that? $\binom{7}{3}$. This generalizes. $\square$