Problem L

The purpose of this problem is to get from the stress-strain relationship

$$\boldsymbol{\Sigma} = (\alpha - \beta)e\mathbf{1} + \beta\mathbf{E}$$

to the result

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} = \left(B_M + \tfrac{1}{3}S_M\right)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + S_M \nabla^2 \mathbf{u}$$

Note: the matrix elements of $\boldsymbol{\Sigma}$ will be written as σ_{ij} and those of $\mathbf{E}$ as ϵ_{ij}. Please show you work — the results are listed in the reading, but I want to see the intermediate steps filled in.

(a) Derive expressions for ϵ_{ij} and e in terms of the distortion field $\mathbf{u}$.

(b) Plug these into the stress-strain equation (in matrix element form) to show

$$\sigma_{ij} = \tfrac{1}{2}(\alpha - \beta)\delta_{ij}\boldsymbol{\nabla} \cdot \mathbf{u} + \tfrac{1}{2}\beta\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$

(c) Derive the following relations:

(i) $\displaystyle\sum_j \delta_{ij} a_j = a_i$

(ii) $\displaystyle\sum_j \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} u_j = [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u})]_i$

(iii) $\displaystyle\sum_j \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_j} u_i = [\nabla^2 \mathbf{u}]_i$

(d) Plug these into the relation

$$[\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}]_i = \sum_j \frac{\partial}{\partial r_j}\sigma_{ji}$$

and substitute $\alpha = 3B_M$ and $\beta = 2S_M$ to derive the final result for $\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}$.