Bucknell University

Theory of Computation
CSCI 341, Fall 2016

Homework 1
Due 2016-09-05

(2pt) Problem 1

What are these sets? Write them using braces, commas, and numerals only.

1. $(\{1,3,5\} \cup \{3,1\}) \cap \{3,5,7\}$
2. $\bigcup\{\ \{3\}, \{3,5\}, \bigcap\{\{5,7\}, \{7,9\}\}\ \}$
3. $(\{1,2,5\} - \{5,7,9\}) \cup (\{5,7,9\} - \{1,2,5\})$
4. $2^{\{7,8,9\}} - 2^{\{7,9\}}$
5. $2^{\emptyset}$

SOLUTION

(a) $(\{1,3,5\} \cup \{3,1\}) \cap \{3,5,7\} = \{1,3,5\} \cap \{3,5,7\} = \{3,5\}$

(b) $\bigcup\{\{3\}, \{3,5\}, \bigcap\{\{5,7\}, \{7,9\}\}\} = \bigcup\{\{3\}, \{3,5\}, \{7\}\} = \{3,5,7\}$

(c) $(\{1,2,5\} - \{5,7,9\}) \cup (\{5,7,9\} - \{1,2,5\}) = \{1,2\} \cup \{7,9\} = \{1,2,7,9\}$

(d) $2^{\{7,8,9\}} - 2^{\{7,9\}} = \{\emptyset, \{7\}, \{8\}, \{9\}, \{7,8\}, \{7,9\}, \{8,9\}, \{7,8,9\}\} - \{\emptyset, \{7\}, \{9\}, \{7,9\}\} =$
$\{\{8\}, \{7,8\}, \{8,9\}, \{7,8,9\}\}$

(e) $2^{\emptyset} = \{\emptyset\}$

(2pt) Problem 2

What are these sets? Write them using braces, parentheses, commas, and numerals only.

1. $\{1\} \times \{1,2\} \times \{1,2,3\}$
2. $\emptyset \times \{1,2\}$
3. $2^{\{1,2\}} \times \{1,2\}$

SOLUTION

(a) $\{1\} \times \{1,2\} \times \{1,2,3\} = \{(1,1,1), (1,1,2), (1,1,3), (1,2,1), (1,2,2), (1,2,3)\}$

(b) $\emptyset \times \{1,2\} = \emptyset$

(c) $2^{\{1,2\}} \times \{1,2\} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \times \{1,2\} =$
$\{(\emptyset,1), (\emptyset,2), (\{1\},1), (\{1\},2), (\{2\},1), (\{2\},2), (\{1,2\},1), (\{1,2\},2)\}$

(2pt) Problem 3

Let $R = \{(a,b),(a,c),(c,d),(a,a),(b,a)\}$. What is $R \circ R$, the composition of R with itself? What is R^{-1}, the inverse of R? Is R, $R \circ R$, or R^{-1} a function?

SOLUTION

We consider the relation $R = \{(a,b),(a,c),(c,d),(a,a),(b,a)\}$. In this case the relation $R \circ R$ and R^{-1} are:

$$R \circ R = \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c)\}$$
$$R^{-1} = \{(b,a),(c,a),(d,c),(a,a),(a,b)\}$$

R is not a function because (a,a) and (a,b) belongs to R.
Similarly, R^{-1} and $R \circ R$ are not functions. (Another reason why R and $R \circ R$ are not functions is because d is not related to any element.)

(2pt) Problem 4

Let $f : A \to B$. Let R_f be the binary relation on A defined as

$$xR_f y \text{ if and only if } f(x) = f(y).$$

Prove that R_f is an equivalence relation.

SOLUTION

We have to prove that R_f is reflexive, symmetric and transitive.
This is simply because the equality $=$ is itself an equivalence relation.

More precisely,

- Let $a \in A$, we have $f(a) = f(a)$ (by reflexivity of $=$), so $(a,a) \in R_f$. We conclude that R_f is reflexive.
- Let $(a,b) \in R_f$, by definition of R_f we have $f(a) = f(b)$, then $f(b) = f(a)$ (by symmetry of $=$), so $(b,a) \in R_f$. We conclude that R_f is symmetric.
- Let $(a,b) \in R_f$ and $(b,c) \in R_f$ then by definition of R_f we have $f(a) = f(b)$ and $f(b) = f(c)$, and then $f(a) = f(c)$ (by transitivity of $=$), and then $(a,c) \in R_f$. We conclude that R_f is transitive.

The relation R_f is an equivalence relation because it is reflexive, symmetric and transitive.

(2pt) Problem 5

Let A be a non-empty finite set and let $f : A \to A$. We have seen the definition of a *cycle* in a relation R. The function f can be seen as its corresponding relation R_f. Prove that R_f contains a cycle. *(Hint: your proof can be the description of an algorithm building such a cycle, together with a clear explanation of the algorithm's correctness.)*

SOLUTION

Let A be a finite set. The cardinality of A is $|A|$. Let $f : A \rightarrow A$ be any function from A to A, we define first the function f^n for $n \in \mathbb{N}$ as $f^0 = id$ and $f^n = f \circ f^n$, with id being the identity function $f(x) = x$. For any number $n \in \mathbb{N}$ and for any $a \in A$, we define the following tuple of length $n+1$:

$$l = (f^0(a), f^1(a), \cdots, f^n(a))$$

The tuple l is a path according to the definition of f^i with $i \leq n$.

Let us fix n being strictly greater than $|A|$. The corresponding path l contains a number of elements greater than $|A|$, then there are necessarily two identical elements. Let us identify these elements by their positions i and j corresponding to the exponents of f, thus we have $f^i(a) = f^j(a)$. We have found a path $l_1 = (f^i(a), \cdots, f^{j-1}(a))$, this path can be seen as a *loop* (which is different from a *cycle*, since duplicates might occur). We still have to prove that this loop contains a cycle. You can prove it in two different ways:

Proof. 1. Construction of a cycle. Let S be the following set $S = \{(m,n) : i \leq m < n \leq j, \text{ and } f^m(a) = f^n(a)\}$. The set S is finite and S defines the set of indices describing loops included in l_1. Let d be the minimum of the difference $n - m$ whenever $(m,n) \in S$, that is written $d = Min\{n - m : (m,n) \in S\}$. The number d exists because is S is finite. Let S' be the set $\{(m,n) : (m,n) \in S \text{ and } n - m = d\}$, S' defines the set of indices of the smallest loops included in l_1. And finally let $(p,q) \in S'$ be the pair such that p is the least number in pairs of S'. By construction, $(f^p(a), \cdots, f^{q-1}(a))$ is a loop containing no loop inside, so it is a cycle. □

Proof. 2. Proof by strong induction that

$P(n)$: any loop of size n contains a cycle.

We have to prove the base case and the inductive case.

- (base case) a loop of size 1 is a cycle.
- (inductive case) Assume the induction hypothesis H : any loop of size less than n contains a cycle. We have to prove that any loop of size $n+1$ contains a cycle. Let L be a loop of size $n+1$, $L = (x_1, x_2, \cdots, x_{n+1})$. There are two cases:
  - all the elements in L are distinct, by definition L is a cycle.
  - two elements in L are equal, assume they are x_i and x_j, we have $x_i = x_j$. In this case $(x_i, \cdots, x_{j-1})$ is a sub-loop of L of size strictly less than $n+1$. We can apply the induction hypothesis H, we conclude that $(x_i, \cdots, x_{j-1})$ contains a cycle and finally L contains a cycle.

  We conclude that in any case L contains a cycle.

Since the base case and the induction case are true we conclude that $P(n)$ is true for all $n \geq 1$. □