ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Review Sheet for Exam #2

The exam is scheduled to take place **3:00–4:50 pm on Friday, November 13 in Academic West 204**. The exam will be designed to be completed in about 1 hour even though you will have 2 hours available.

The following policies will be in effect for the exam:

1. You will be allowed to use a non-wireless enabled calculator, such as a TI-99. You must put away smart watches, cell phones, tablets, laptops, and other devices that can connect to the internet wirelessly. I will record your calculator make and model during the exam.
2. You will be allowed to use up to 10 two-sided paper help sheets. They may include the notes that you recorded in class (original or copied), homework solutions, supplemental notes from the course Moodle site, copied or printed pages from the textbook, or any other printed or handwritten material. Hard copies of the textbook will not be allowed since not every student has access to one. If there is anything else that you feel would be helpful, please notify me.
3. If you arrive late to the exam, you must complete it in the remaining allotted time. However, you will not be allowed to take the exam if you arrive after the first student has completed it and left the room. The latter case is equivalent to missing the exam. As stated in the syllabus, an unexcused absence from an exam will result in a 5-point exam score reduction.
4. **You may not leave the exam room without prior permission except in an emergency or for an urgent medical condition. Please use the restroom before the exam.**

The following is a list of topics that could appear in one form or another on the exam. Not all of these topics will be covered, and it is possible that an exam problem could cover a detail not specifically listed here. However, this list has been made as comprehensive as possible. You should be familiar with the topics on the previous review sheet in addition to those listed below.

Although significant effort has been made to ensure the accuracy of this review sheet, some errors might nevertheless appear. The textbook, the course lecture notes, and the supplemental readings are the final factual authorities unless errors have been specifically identified there. You are ultimately responsible for obtaining accurate information when preparing for the exam.

Ordinary differential equations (ODEs) – general properties and theorems

- linear differential equations have the form

  $$a_n(x)\frac{d^n y}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x)\,y = g(x)$$

  coefficients can be constant or variable; if variable, they are functions of indep. variable
- important special case is the 2nd order linear equation
- distinction between homogeneous and inhomogeneous ODEs; does $g(x) = 0$?
- distinction between initial value problems (IVPs) and boundary value problems (BVPs)
  - IVP: dependent variable and/or its derivatives defined at one point
  - IVP: always have unique solutions
  - BVP: dependent variable and/or its derivatives defined at two or more points
  - BVP: can have no solution, a unique solution, or many solutions
  - BVPs are often eigenvalue problems (EVPs) as well

- superposition principle; a linear combination of solutions to a homogeneous ODE over an interval is also a solution over the same interval
- corollaries:
  - a constant multiple of a solution is also a solution
  - a homogeneous ODE always possess the trivial solution $y = 0$
- linearly independent vs. linearly dependent solutions (analogy to vectors)
- N^{th} order homogeneous linear ODE has a fundamental set of N linear indep. solutions
- nonhomogeneous ODEs: general solution = complementary solution + particular solution (complementary solution is the full solution set of the corresponding homogenous ODE)
- superposition also applies to particular solutions: If y_{p1} is a solution of the ODE with $g_1(x)$, y_{p2} is a solution with $g_2(x)$, etc., then $y_{p1} + y_{p2} + \ldots$ is a solution to ODE with $g(x) = g_1(x) + g_1(x) + \ldots$
- reduction of order method to find second linearly independent solution

  $$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx,$$

  where $P(x)$ is the variable coefficient in the general form $y'' + P(x)y' + Q(x)y = 0$

Fourier and modified Fourier equations and their suggested solutions

- Fourier equation: $y'' + a^2 y = 0$

  suggested solution for closed boundaries: $y(x) = c_1\cos(ax) + c_2\sin(ax)$
- modified Fourier equation: $y'' - a^2 y = 0$

  suggested solution for closed boundaries: $y(x) = c_1\cosh(ax) + c_2\sinh(ax)$
- Fourier equation w/homogeneous boundary conditions (BCs) are eigenvalue problems (EVPs)
- modified Fourier equation w/homogeneous BCs has only the trivial solution
- Fourier equation and modified Fourier equation w/nonhomogeneous BCs can have nontrivial solutions, but they might not be eigenvalue problems

Power series solutions to ODEs

- constant coefficients → find roots of characteristic equation; power series not necessary
- variable coefficients → power series solution

  $$y(x) = \sum_{n=0}^{\infty} c_n x^n$$
- power series approach works in general for any ODE but is unnecessary for constant coefficients and many forms of the Cauchy-Euler equation
- ordinary points: values of x at which variable coefficients are analytic
- singular points: values of x that are not ordinary points
  example: $\ln(x)$ is not analytic for $x \le 0$, so the ordinary points are $x > 0$
- power series solutions valid only over intervals for which they converge; often not valid over all of $-\infty < x < \infty$

Frequently encountered 2nd-order ODEs and their suggested solutions (there are many other ODEs besides these); most solutions are those recommended for closed boundaries

- Fourier equation: $y'' + a^2 y = 0$

  solution: $y(x) = c_1\cos(ax) + c_2\sin(ax)$
- modified Fourier equation: $y'' - a^2 y = 0$

  solution: $y(x) = c_1\cosh(ax) + c_2\sinh(ax)$

- 2nd order Cauchy-Euler equation: $ax^2y'' + bxy' + cy = 0$
  - try solutions of the form $y = x^m$
  - find roots of auxiliary eqn $am(m-1) + bm + c = 0 \quad \rightarrow \quad am^2 + (b-a)m + c = 0$
  - for distinct real roots m_1 and m_2, $y(x) = c_1 x^{m_1} + c_2 x^{m_2}$
  - for repeated real roots m_1, $y(x) = c_1 x^{m_1} + c_2 x^{m_1} \ln x$
  - for complex conjugate roots $m = \alpha \pm i\beta$,
    $y(x) = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$
  - special case: $x^2y'' + xy' - \alpha^2 y = 0$

    solutions are $y = \begin{cases} c_1 + c_2 \ln x, & \alpha = 0 \\ c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha > 0 \end{cases}$
  - further special case: $x^2y'' + xy' - y = 0$ $(\alpha = 1)$

    solution is $y = \dfrac{c_1}{x} + c_2 x$
- Bessel equation: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ (ν often replaced with n)

  solution: $y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$ (Bessel functions of the 1st & 2nd kind)
- modified Bessel equation: $x^2y'' + xy' - (x^2 + \nu^2)y = 0$ (ν often replaced with n)

  solution: $y(x) = c_1 I_\nu(x) + c_2 K_\nu(x)$ (modified Bessel functions of the 1st & 2nd kind)
- parametric Bessel equation: $x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0$

  solution: $y(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$
- modified parametric Bessel equation: $x^2y'' + xy' - (\alpha^2x^2 + \nu^2)y = 0$

  solution: $y(x) = c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x)$

Inner product of two functions
- analogous to vector inner product
- definition:
  given two functions $y_m(x)$ and $y_n(x)$ and a weighting function $p(x)$,
  $\langle y_m, y_n \rangle_{p(x)} = \int_a^b y_m(x)\, y_n(x) p(x)\, dx$,
  where a and b specify the interval over which the inner product is defined
- norm: $\|y_n\| = \sqrt{\int_a^b y_n^2(x)\, p(x)\, dx}$; square norm (self-product): $\|y_n\|^2 = \int_a^b y_n^2(x)\, p(x)\, dx$

Boundary conditions
- Dirichlet: conditions on dependent variable [e.g., $y(L) = 0$]
- Neumann: conditions on derivative(s) of dependent variable [e.g., $y'(L) = 0$]
- Robin: conditions on dependent variable and derivative(s) combined
  [e.g., $A_1\, y(L) + B_1\, y'(L) = C_1$]
- definition of homogeneous BVP: right-hand side of ODE *and* BCs = 0

Regular Sturm-Liouville problem

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right]+q(x)y+\lambda p(x)y=0 \quad \text{(self-adjoint form)}$$

- some textbooks use $p(x)$ in place of $r(x)$ and $w(x)$ in place of $p(x)$
- second-order with variable coefficients
- self-adjoint form allows identification of weight function $p(x)$ used in inner product
- λ is a parameter in the problem (eigenvalue)
- must have $r(x)$, $p(x) > 0$ over interval of solution (a, b) for a *regular* S-L problem
- boundary conditions defined over interval (a, b):

$$A_1 y(a)+B_1 y'(a)=0$$

$$A_2 y(b)+B_2 y'(b)=0$$

Conversion of second-order ODE to self-adjoint Sturm-Liouville form (not always possible):

$$a(x)y''+b(x)y'+c(x)y+\lambda d(x)y=0 \quad \rightarrow \quad \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right]+q(x)y+\lambda p(x)y=0$$

a. compute the integrating factor $\mu(x)$ (watch out for $a(x) = 0$ for any x over the bounded interval):

$$\mu(x)=\exp\left(\int\frac{b(x)}{a(x)}dx\right)$$

b. compute the elements of the S-L self-adjoint form:

$$r(x)=\mu(x) \qquad q(x)=\frac{c(x)}{a(x)}\mu(x) \qquad p(x)=\frac{d(x)}{a(x)}\mu(x)$$

c. verify that $r(x)$, $p(x) > 0$ over interval of solution for regular S-L problem

Orthogonality conditions on solutions to Sturm-Liouville problem

- relationship between inner product and boundary conditions (BCs)

$$(\lambda_m-\lambda_n)\int_a^b p(x)y_m(x)y_n(x)dx = r(b)\left[y_m(b)y_n'(b)-y_n(b)y_m'(b)\right] - r(a)\left[y_m(a)y_n'(a)-y_n(a)y_m'(a)\right]$$

- implications:
  - BCs must be homogenous if the solutions y_m and y_n are to be orthogonal. If $m \neq n$ and the BCs are homogeneous, then the right-hand side equals zero
  - y_m and y_n can be orthogonal if $r(x) = 0$ at one of the boundaries and the BC at the other boundary is homogeneous or if $r(a) = r(b) = 0$

Singular Sturm-Liouville problem

- addresses case when $r(x) > 0$ is not satisfied at one or both boundaries
- right-hand side of equation in inner product relationship above is zero when:
  - $r(a) = 0$ and $y_m(b)y_n'(b)-y_n(b)y_m'(b)=0$
  - $r(b) = 0$ and $y_m(a)y_n'(a)-y_n(a)y_m'(a)=0$
  - $r(a) = r(b) = 0$ and no BCs are specified at $x = a$ or $x = b$
  - $r(a) = r(b)$ and the BCs are $y(a) = y(b)$ and $y'(a) = y'(b)$ (periodic BCs)
  - the solutions $\{y_n\}$ are orthogonal if $r(a) = 0$ and/or $r(b) = 0$ provided that the solutions are bounded (i.e., do not approach ∞) at the corresponding boundary

The "Sturm-Liouville Insurance Policy" (SLIP)

a. There are non-trivial solutions for specific values of the parameter λ (eigenvalues).
b. There is an infinity of eigenvalues.
c. There is a smallest but not a largest eigenvalue.
d. The eigenvalues are real and distinct ($\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$).
e. For each eigenvalue there is a single solution (eigenfunction) $y_n(x)$
f. The eigenfunctions corresponding to two different eigenvalues are orthogonal with respect to the weight function $p(x)$. That is, their inner product is

$$\langle y_m, y_n \rangle_{p(x)} = \int_a^b y_m(x)\, y_n(x) p(x)\, dx = \begin{cases} 0, & m \neq n \\ C_m, & m = n \end{cases}$$

where C_m is some non-zero constant

g. The set of solutions is complete; it forms a basis for the space of square-integrable functions on the interval [a, b]:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x), \quad \text{where} \quad a_n = \frac{\langle f(x), y_n(x) \rangle}{\langle y_n(x), y_n(x) \rangle} = \frac{\langle f(x), y_n(x) \rangle}{\| y_n(x) \|^2}$$

Solution of partial differential equations (PDEs) using the separation of variables (SoV) method

- not always possible to find a solution with this method; some PDE solutions are not separable
- overall solution assumed to be product of solutions in each independent variable
- solution of ODE in one independent variable could be an infinite sum of eigenfunctions corresponding to different eigenvalues (superposition principle)
- focus on three commonly encountered homogeneous equations (variations are possible):
  - heat equation: $k \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k \geq 0$
  - wave equation: $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$
  - Laplace's equation: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$
- boundary conditions:
  - Dirichlet – apply to dependent variable at boundaries
  - Neumann – apply to spatial derivative of dependent variable at boundaries
  - Robin – relationship b/w dependent variable and its spatial derivative
  - open – no physical boundaries; solution space extends indefinitely
- typical initial conditions:
  - $u(x, 0) = f(x)$
  - $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = g(x)$
- separated 2nd-order ODEs could have eigenvalues that satisfy $\lambda < 0$, $\lambda = 0$, and/or $\lambda > 0$; form of solution depends on which one(s) lead(s) to non-trivial solutions; in that case:
  - solution usually takes form of infinite series of eigenfunctions in all independent variables
  - coefficients in infinite series are determined using inner products of orthogonal eigenfunctions

- interpretation of solutions
    - determine whether solution is correct (does it make sense?)
    - heat/diffusion problems – does time progression of solution correspond to heat/concentration gradients?
    - wave equation – solutions are typically standing waves at harmonically related frequencies (in Cartesian coordinate system) if solution space is bounded; equivalent to superpositions of counterpropagating traveling waves
    - Laplace's equation – solution must satisfy the maximum principle (max and min values on boundary; no local max/min extrema in interior of solution space) [not covered on this exam]
- d'Alembert's solution for wave equation with open boundary conditions

Relevant course material:

Homework: #4, #5, and #6
Labs: #3, #4, and #5
Reading: Assignments from Sept. 28 through Nov. 4, including the supplemental readings:
"Series Solution Method for ODEs"
"Meet the Bessels"
"Sturm-Liouville Problems: Eigenfunction Orthogonality"
"D'Alembert's Solution of the 1-D Wave Equation Problem in Unbounded Media"

This exam will focus primarily on the course outcomes listed below, although only the parts indicated in brackets will be covered.

- Develop the concept of an eigenvalue problem in both linear algebra and differential equations. [differential equations only]
- Introduce important analytical and numerical methods for the solution of PDEs. [analytical only]

The course outcomes are listed on the course syllabus, which was distributed at the beginning of the semester and is available on the Syllabus and Policies page at the course web site. The outcomes are also listed on the Course Description page at the web site. Note, however, that some topics not directly related to the course outcomes could be covered on the exam as well.