

Homework Assignment #4 – due via Moodle at 11:59 pm on Friday, Oct. 23, 2026***Instructions, notes, and hints:***

Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the posted solutions and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Submit only the graded problems by the deadline above. Do not submit the ungraded problems.

Graded Problems:

1. Shown below is the standard Fourier equation. A pair of linearly independent solutions is $\sin(ax)$ and $\cos(ax)$. Suppose that you only knew that $\sin(ax)$ was a solution. Use Equation (5) in Section 3.2 of the textbook, which is based on the reduction of order method, to prove that $\cos(ax)$ (perhaps multiplied by a constant) is a second linearly independent solution. You may consult a table of integrals; there is a link to multiple tables and lists of more tables on the Web Links page at the course web site.

$$y'' + a^2 y = 0, \quad a > 0$$

2. Find the eigenvalues and eigenfunctions of the boundary value problem (BVP) given below. Also specify the form of the inner product used to test the orthogonality of the solutions $\{y_n(x)\}$.

$$x^2 y'' + xy' + \lambda y = 0 \quad \text{with } \lambda = \alpha^2, \alpha > 0 \text{ and BCs } y(1) = 0 \text{ and } y(5) = 0$$

3. Show that the BVP in the previous problem has only the trivial solution if $\lambda < 0$.
4. Show that a solution to the boundary value problem given below with the indicated boundary conditions is the expression for $y(x)$ given farther below with the indicated eigenvalues.

$$y'' + a^2 y = 0, \quad a > 0 \quad \text{with} \quad y(\pi/4) = 0 \text{ and } y(\pi) = 0$$

$$y(x) = C \left[-\tan(a_n \pi) \cos(a_n x) + \sin(a_n x) \right], \text{ where } C \text{ is a constant and } a_n = \frac{4}{3}n; \quad n = 1, 2, 3, \dots$$

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Ungraded Problems:

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. Shown that the function xe^{-5x} is a solution of the ODE shown below. Note that the auxiliary equation is $r^2 + 10r + 25 = 0$, which has the repeated root $r = -5$.

$$y'' + 10y' + 25y = 0$$

2. Find solutions to the following boundary value problems (BVPs). Not all are eigenvalue problems (EVPs).

a. $x^2 y'' + xy' + a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 1$

b. $x^2 y'' + xy' - a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 1$

c. $x^2 y'' + xy' + a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 0$

d. $x^2 y'' + xy' - a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 0$

3. The norm and square norm of a function are defined by the expressions given below left. The square norm is also the self-product of a function. Find the square norm of the Fourier eigenfunction $y_n(x) = \cos(n\pi x)$, $n = 0, 1, 2, \dots$ over the interval $[0, 1]$ using analytical means. Provide a general result that applies for all values of n . You may look up the solution in an integral table to check your answer, but you must show the solution. *Hint:* Use a trigonometric identity or the identity given below right.

norm: $\|y_n\| = \sqrt{\int_a^b y_n^2(x) dx}$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

square norm: $\|y_n\|^2 = \int_a^b y_n^2(x) dx$

4. Chebyshev's equation is given below. For appropriate boundary conditions, it leads to an eigenvalue problem with orthogonal solutions $\{T_n(x)\}$ for $n = 0, 1, 2, \dots$ that correspond to the eigenvalues $\{\lambda_n\}$. The solutions take the form of polynomials of order n ; that is, as n increases, so does the order of the polynomial. Find the interval of x over which the solutions are orthogonal, and specify the form of the inner product used to test the orthogonality of the solutions $\{T_n(x)\}$.

$$(1 - x^2)y'' - xy' + n^2y = 0$$

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5. The textbook (Zill, 6th ed.) presents a proof for the orthogonality of the nontrivial solutions to a Sturm-Liouville problem by starting with the self-adjoint form for two different eigenvalues and eigenfunctions given by

$$\frac{d}{dx} \left[r(x) \frac{dy_m}{dx} \right] + q(x) y_m + \lambda_m p(x) y_m = 0$$

$$\frac{d}{dx} \left[r(x) \frac{dy_n}{dx} \right] + q(x) y_n + \lambda_n p(x) y_n = 0$$

and then stating that multiplying the first equation by y_n and the second by y_m , subtracting the two equations, and finally integrating by parts from $x = a$ to $x = b$ yields the expression

$$(\lambda_m - \lambda_n) \int_a^b p(x) y_m(x) y_n(x) dx = r(b) [y_m(b) y_n'(b) - y_n(b) y_m'(b)] \\ - r(a) [y_m(a) y_n'(a) - y_n(a) y_m'(a)].$$

Work through the missing steps to prove the orthogonality condition above.