ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Homework Assignment #5 – due via Moodle at 11:59 pm on Thursday, Nov. 5, 2026

***Instructions, notes, and hints*:**

Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the posted solutions and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Submit only the graded problems by the deadline above. Do not submit the ungraded problems.

***Graded Problems*:**

1. Find solutions to the Fourier equation $y'' + a^2 y = 0$, with $a > 0$, for the following sets of boundary conditions. After you complete the solutions, determine whether each problem is an eigenvalue problem and whether it has a nontrivial solution. Relate your answers to whether the boundary conditions are homogeneous.

a. $y(0) = 1$ and $y(4) = 0$
b. $y(0) = 0$ and $y(4) = 0$

2. Find solutions to the modified Fourier equation $y'' - a^2 y = 0$, with $a > 0$, for the following sets of boundary conditions. After you complete the solutions, determine whether each problem is an eigenvalue problem and whether it has a nontrivial solution. Relate your answers to whether the boundary conditions are homogeneous.

a. $y(0) = 1$ and $y(4) = 0$
b. $y(0) = 0$ and $y(4) = 0$

3. Find the eigenvalues and eigenfunctions of the boundary value problem (BVP) given below. Also specify the form of the inner product used to test the orthogonality of the solutions $\{y_n(x)\}$.

$$x^2 y'' + xy' + \lambda y = 0 \text{ with } \lambda = \alpha^2,\ \alpha > 0 \text{ and BCs } y(1) = 0 \text{ and } y(5) = 0$$

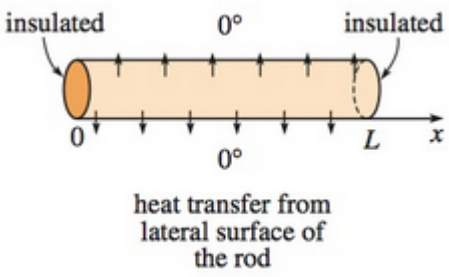
4. Show that the BVP in the previous problem has only the trivial solution if $\lambda < 0$, that is, if $\lambda = -\alpha^2$.

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5. [adapted from Prob. 5 in Sec. 13.3 of D. G. Zill, 6th ed.; the diagram below is from the same source] Heat is lost from the lateral surface of a thin rod of length L into a surrounding medium that has a temperature of 0 K. If the linear law of heat transfer applies, then the heat equation has the form

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \qquad 0 \le x \le L \quad \text{and} \quad t \ge 0$$

where h is a constant. Find an expression for the temperature $u(x,t)$ if the initial temperature is $f(x)$ over the interval $[0, L]$ and the ends $x = 0$ and $x = L$ are insulated as shown in the figure to the right. *Hint*: The insulation defines the boundary conditions.



6. **[Graduate students only]** Attempt to use the separation of variables method to decompose the following partial differential equations into a set of ordinary differential equations (ODEs). For the problem that involves more than two independent variables, you will need more than one separation constant. You do not have to find the solutions to the ODEs.

a. $\dfrac{\partial^2 E}{\partial x^2} + \dfrac{\partial^2 E}{\partial y^2} + \dfrac{\partial^2 E}{\partial z^2} = 0$ (Use separation constants k_x^2, k_y^2, and k_z^2)

b. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

c. $y\dfrac{\partial^2 u}{\partial x\,\partial y} + u = 0$

d. $k\dfrac{\partial^2 u}{\partial x^2} - u = \dfrac{\partial u}{\partial t}$, where k is a constant and $k > 0$

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. Determine whether the following problems are initial value problems (IVPs), boundary value (BVPs), or neither. For both differential equations, $a > 0$. Briefly explain your answer for each part.

a. $y'' + a^2 y = 0$ with $y(0) = 1$ and $y(1) = 0$

b. $y'' + a^2 y = 0$ with $y(0) = 0$ and $y'(0) = 1$

2. Find the solutions of the BVP given below. Consider the conditions $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. Determine whether each case leads to an eigenvalue problem and whether it has a nontrivial solution.

$$y'' + \lambda y = 0 \text{ with } y(0) = 0 \text{ and } y'(\pi/2) = 0$$

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3. Find the solution to the BVP given below. Note that the boundary conditions are not homogeneous. Consider the conditions $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

$$y'' + \lambda y = 0 \text{ with } y(0) = 1 \text{ and } y'(\pi/2) = 0$$

4. Find the eigenvalues and eigenfunctions of the BVP given below. Consider the conditions $\lambda < -1$, $\lambda = -1$, and $\lambda > -1$.

$$y'' + (\lambda + 1) y = 0 \text{ with } y'(0) = 0 \text{ and } y'(1) = 0$$

5. Find a SoV solution to the 1-D heat equation given below left, where u = temperature, for a rod of length $L = 2$ m and insulated ends. The solution must include the numerical values of the set of coefficients $\{A_n\}$, which has only two nonzero values. The initial temperature distribution $f(x)$ along the rod is given below. The thermal diffusivity of the rod is $k = 0.01$ m^2/s. Use the hint below right to show that the coefficients for $n = 3$ and higher are all zero. You may use mathematical analysis software such as *Matlab* or *Mathematica* or a calculator to perform the numerical integrations required to obtain the two coefficient values.

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad f(x) = 200 \sin^2\left(\frac{\pi x}{L}\right) \qquad \textit{Hint: } \sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

6. [adapted from Prob. 30 in Sec. 3.9 of Zill, 6th ed.] The temperature distribution $u(r)$ in a circular ring or annulus is determined from the BVP given by

$$r \frac{d^2 u}{dr^2} + \frac{du}{dr} = 0 \text{ with } u(a) = u_0 \text{ and } u(b) = u_1,$$

where u_0 and u_1 are constants. Show that

$$u(r) = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(a/b)}$$

7. Attempt to use the separation of variables method to find a solution to the following partial differential equation.

$$\frac{\partial^2 u}{\partial x \, \partial y} + \frac{\partial u}{\partial x} = 0$$

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8. Recall that the homogeneous boundary value problem involving the one-dimensional heat equation and its boundary conditions and initial condition can be expressed as

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \qquad 0\le x\le L \quad \text{and} \quad t\ge 0$$

$$\text{with } u(0,t)=0 \quad u(L,t)=0 \quad u(x,0)=f(x),$$

where the dependent variable u represents the relative heat or temperature. The function $f(x)$ specifies the heat or temperature distribution along the spatial interval $[0, L]$ at time $t = 0$. Applying the SoV method and assuming that $u(x, t) = X(x)T(t)$ yields an expression of the form

$$\frac{X''}{X}=\frac{1}{k}\frac{T'}{T}=-\lambda,$$

where $-\lambda$ is the separation constant. The resulting set of ordinary differential equations (ODEs) is

$$X''+\lambda X=0 \qquad \text{and} \qquad T'+k\lambda T=0,$$

and the resulting solution is

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t/L^2}\sin\left(\frac{n\pi}{L}x\right), \quad \text{where} \quad A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Note that the thermal diffusivity k could have been associated with the ODE in $X(x)$ instead of the ODE in $T(t)$. That is, the application of the SoV method could have led to the alternate expression

$$k\frac{X''}{X}=\frac{T'}{T}=-\lambda.$$

Show that the solution for $u(x, t)$ given above is still obtained when the alternate expression is used as the starting point.