# ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

**Homework Assignment #6 – due via Moodle at 11:59 pm on Tuesday, Nov. 10, 2026**

***Instructions, notes, and hints*:**

Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the posted solutions and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Submit only the graded problems by the deadline above. Do not submit the ungraded problems.

***Graded Problems*:**

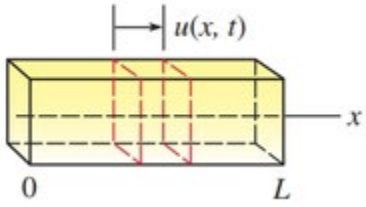
**1.** [adapted from Prob. 15 in Sec. 13.4 of D. G. Zill, 6th ed.] A string is stretched and secured on the $x$-axis at $x = 0$ and $x = \pi$ for $t > 0$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the governing wave equation has the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t}, \qquad 0 < \beta < 1, \quad t > 0, \quad 0 < x < \pi .$$

Find an expression for the displacement $u(x,t)$ if the string starts from rest from the initial displacement $f(x)$. A string that starts from rest has $g(x) = 0$.

**2.** [adapted from Prob. 11 in Sec. 13.4 of D. G. Zill, 6th ed.] The longitudinal displacement of the vibrating elastic bar illustrated below (Fig. 13.4.7 from D. G. Zill, 6th ed.) satisfies the wave equation with the boundary conditions and initial conditions given below. Find the displacement $u(x, t)$. The bar is initially at rest, so $g(x) = 0$ and therefore the initial condition on $\partial u/\partial t$ is zero along the bar.

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad t > 0, \quad 0 < x < L$$




$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \quad u(x,0) = x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

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**3.** The transverse displacement of a vibrating string satisfies the wave equation with the boundary conditions and initial conditions given below. As we have seen earlier, $v$ is the velocity of waves along the string. The initial conditions simulate the arrival of a sound wave at the lowest resonant frequency of the string at time $t = 0$. Find as complete a solution of the displacement $u(x, t)$ as possible and comment on the significance of the solution.

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad t > 0, \quad 0 < x < L$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.2 \sin\left(\frac{\pi x}{L}\right)$$

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

**1.** D'Alembert's solution to the one-dimensional wave equation in unbounded media, defined by

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad -\infty \le x \le \infty \quad \text{and} \quad t \ge 0$$

$$u(x, 0) = f(x) \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x)$$

is given by

$$u(x,t) = \frac{1}{2} f(x + at) + \frac{1}{2} f(x - at) + \frac{1}{2a} \int_{x-at}^{x+at} g(s)\, ds \,.$$

Consider a vibrating string problem in which the dependent variable $u$ represents the displacement of the string from its equilibrium position, and $a = 300$ m/s, $f(x) = 0$, and $g(x)$ is a rectangular pulse function given by

$$g(x) = \text{Rect}(x) = \begin{cases} A, & -0.25 \text{ m} \le x \le 0.25 \text{ m} \\ 0, & \text{elsewhere} \end{cases}$$

where $A = 900$ m/s. This simulates what would happen if someone were to strike the string with a 50 cm wide flat object. Plot the solution $u(x, t)$ over the spatial interval $x = -2$ m to $x = 2$ m at the times $t = 0.5$, 1, 2, and 4 ms. You will probably want to use mathematical analysis software like *Matlab* or *Mathematica* to produce the plots.

*Hint*: Consider evaluating the integral in the solution for six different cases ($r = 0.25$ m, the half-width of the object that strikes the string):

| | |
|---|---|
| $x - at < -r$ and $x + at < -r$ | $-r < (x - at) < r$ and $-r < (x + at) < r$ |
| $x - at < -r$ and $-r < (x + at) < r$ | $-r < (x - at) < r$ and $x + at > r$ |
| $x - at < -r$ and $x + at > r$ | $x - at > r$ and $x + at > r$ |

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**2.** The homogeneous boundary value problem involving the one-dimensional heat equation and its boundary conditions and initial condition can be expressed as

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \qquad 0\le x\le L \quad \text{and} \quad t\ge 0$$

$$\text{with } u(0,t)=0 \quad u(L,t)=0 \quad u(x,0)=f(x),$$

where the dependent variable $u$ represents the relative heat or temperature. The function $f(x)$ specifies the heat or temperature distribution along the spatial interval $[0, L]$ at time $t = 0$. Applying the SoV method and assuming that $u(x, t) = X(x)T(t)$ yields an expression of the form

$$\frac{X''}{X}=\frac{1}{k}\frac{T'}{T}=-\lambda,$$

In the usual solution approach, the separation constant is $-\lambda$ as shown above. The minus sign is included because experience has shown that it simplifies the analysis. However, the separation expression could have the form

$$\frac{X''}{X}=\frac{1}{k}\frac{T'}{T}=\lambda.$$

Show that the solution to the problem is still

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k\left(n^2\pi^2/L^2\right)t}\sin\left(\frac{n\pi}{L}x\right), \quad \text{where} \quad A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

if the expression with an unnegated separation constant $\lambda$ is used as the starting point.

*Hint*: The usual assumed solution to the ODE $X'' - \lambda X = 0$ is of the form

$$X(x)=c_1\cosh\left(\sqrt{\lambda}\,x\right)+c_2\sinh\left(\sqrt{\lambda}\,x\right) \quad \text{or} \quad X(x)=c_1e^{\sqrt{\lambda}\,x}+c_2e^{-\sqrt{\lambda}\,x}.$$

However, consider that an alternative solution might have the form

$$X(x)=c_1\cos\left(i\sqrt{\lambda}\,x\right)+c_2\sin\left(i\sqrt{\lambda}\,x\right),$$

where $i$ is the square root of $-1$. Other potentially useful formulas:

$$\cosh(x)=\frac{e^x+e^{-x}}{2} \qquad \sinh(x)=\frac{e^x-e^{-x}}{2} \qquad \cos(x)=\frac{e^{ix}+e^{-ix}}{2} \qquad \sin(x)=\frac{e^{ix}-e^{-ix}}{2i}$$