ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Wednesday, Nov. 11, 2026

1. Continue with wave equation in polar (cylindrical) coordinate system example: Vibrating circular membrane of radius c:

$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad 0\le r\le c \quad \text{and} \quad t\ge 0$$

   In the case of a drum being struck by a stick or mallet, $f(r) = 0$ and $g(r)$ is a pulse centered at $r = 0$.

2. SoV solution and its interpretation (x_n = roots of J_0)

$$u(r,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(v\alpha_n t\right)+B_n\sin\left(v\alpha_n t\right)\right]J_0\left(\alpha_n r\right) \quad \text{with} \quad \sqrt{\lambda_n}=\alpha_n=\frac{x_n}{c}$$

   a. Inner product of Bessel functions; weighting function is $p(r) = r$

$$\left\langle J_0\left(\alpha_m r\right),J_0\left(\alpha_n r\right)\right\rangle=\int_0^c r\,J_0\left(\alpha_m r\right)J_0\left(\alpha_n r\right)dr$$

   b. Apply ICs to find expressions for $\{A_n\}$ and $\{B_n\}$ coefficients

$$u(r,0)=f(r)=\sum_{n=1}^{\infty}\left[A_n(1)+B_n(0)\right]J_0\left(\alpha_n r\right)$$

$$\int_0^c r\,f(r)J_0\left(\alpha_m r\right)dr=\sum_{n=1}^{\infty}A_n\int_o^c r\,J_0\left(\alpha_n r\right)J_0\left(\alpha_m r\right)dr \quad\rightarrow\quad A_n=\frac{\left\langle f(r),J_0\left(\alpha_n r\right)\right\rangle}{\left\|J_0\left(\alpha_n r\right)\right\|^2}$$

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left[-A_n v\alpha_n\sin\left(v\alpha_n t\right)+B_n v\alpha_n\cos\left(v\alpha_n t\right)\right]J_0\left(\alpha_n r\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r)=\sum_{n=1}^{\infty}v\alpha_n B_n J_0\left(\alpha_n r\right)$$

$$\int_0^c r g(r)J_0\left(\alpha_m r\right)dr=\sum_{n=1}^{\infty}v\alpha_n B_n\int_0^c J_0\left(\alpha_n r\right)J_0\left(\alpha_m r\right)$$

$$\rightarrow\quad B_n=\frac{\left\langle g(r),J_0\left(\alpha_n r\right)\right\rangle}{v\alpha_n\left\|J_0\left(\alpha_n r\right)\right\|^2}$$

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c. What does the result mean? How are the eigenvalues used and interpreted?

Vibration frequencies: $\omega_n = 2\pi f_n = v\alpha_n = \frac{vx_n}{c} \quad \rightarrow \quad f_n = \frac{vx_n}{2\pi c}$.
where $x_1 = 2.4048$, $x_2 = 5.5201$, $x_3 = 8.6537$, $x_4 = 11.7915$, etc.

Frequencies are proportional to x_n, but…

$$x_2 = 2.29x_1,\ x_3 = 3.59x_1,\ x_4 = 4.89x_1,\ \ldots \text{ (no harmonic relationships)}$$

d. *Matlab* simulation
e. Are there standing waves? How do they compare to the vibrating string case?

3. Next: Numerical solution of PDEs using finite differences

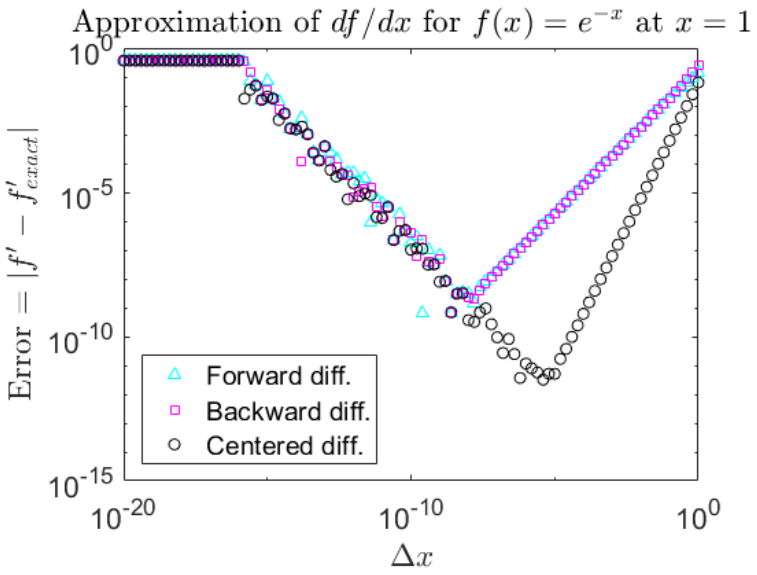
a. First derivative approximations:

i. Forward difference: $\frac{df(x_o)}{dx} \approx \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$

ii. Backward difference: $\frac{df(x_o)}{dx} \approx \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x}$

iii. Centered difference: $\frac{df(x_o)}{dx} \approx \frac{f(x_o + 0.5\Delta x) - f(x_o - 0.5\Delta x)}{\Delta x}$

iv. Error comparison (Δx larger to the right in graph):



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b. Second derivative approximation: $\dfrac{d^2 f(x_o)}{dx^2} \approx \dfrac{f(x_o+\Delta x)-2f(x_o)+f(x_o-\Delta x)}{\Delta x^2}$

c. Example: For $f(x)=e^x$, approximate $f'(1.2)$ using centered finite differences with $\Delta x = 0.1$, 0.05, and 0.01. Exact result (to 5 sig. digs.) is $f'(1.2) = 3.3201$.

i. $\Delta x = 0.1$: $\quad f'(1.2) \approx \dfrac{f(1.2+0.05)-f(1.2-0.05)}{0.1} = \dfrac{e^{1.25}-e^{1.15}}{0.1} = 3.3215$

ii. $\Delta x = 0.05$: $\quad f'(1.2) \approx \dfrac{f(1.2+0.025)-f(1.2-0.025)}{0.05} = \dfrac{e^{1.225}-e^{1.175}}{0.05} = 3.3205$

iii. $\Delta x = 0.01$: $\quad f'(1.2) \approx \dfrac{f(1.2+0.005)-f(1.2-0.005)}{0.01} = \dfrac{e^{1.205}-e^{1.195}}{0.01} = 3.3201$