

Lecture Outline for Wednesday, Oct. 14, 2026

1. Fourier equation as an eigenvalue problem; three possible solution forms (for closed boundaries)

$$y'' + \lambda y = 0$$

- a. If $\lambda < 0$: $y(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x)$
- b. If $\lambda = 0$: $y(x) = c_1 + c_2 x$
- c. If $\lambda > 0$: $y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

2. ODEs with variable coefficients

- a. Solutions to ODEs with constant coefficients are formed using real or complex exponential functions or linear combinations of them (cos/sin or cosh/sinh).
- b. ODEs with variable coefficients require a different approach.
- c. Some (like the Cauchy Euler equation) have a form that permits a particular type of solution approach (example below).
- d. The most general approach is to use a power series (soon).

3. Cauchy-Euler equation: power of monomial coefficients matches order of differentiation

2nd-order examples:

$$ax^2y'' + bxy' + cy = 0$$

special case: $x^2y'' + xy' - \alpha^2y = 0$ ($a = b = 1$ and c neg.)

special special case: $x^2y'' + xy' - y = 0$ ($a = b = 1$ and $c = -1$)

4. “Peel-the-onion” method applied to Cauchy-Euler equation with $a = b = 1$ and $c = -1$

$$x^2y'' + xy' - y = 0 \quad \text{equiv. to} \quad \frac{d}{dx} \left[\frac{1}{x} \frac{d}{dx} (xy) \right] = 0$$

Successive integrations to arrive at solution. First integration w.r.t. x :

$$\frac{1}{x} \frac{d}{dx} (xy) = c_1 \quad \rightarrow \quad \frac{d}{dx} (xy) = c_1 x$$

Second integration w.r.t. x :

$$xy = c_1 \frac{x^2}{2} + c_2 \quad \rightarrow \quad y = c_1 \frac{x}{2} + c_2 \frac{1}{x}$$

(continued on next page)

5. Solution to 2nd order Cauchy-Euler equation with $a = b = 1$ and c neg. (See Sec. 3.6 of Zill, 6th ed.; see also Sec. 3.2, which explains the reduction of order method for finding a second independent solution); attempt solutions of the form $y = x^r$, where r is a constant.

$$x^2 y'' + xy' - \alpha^2 y = 0 \quad \text{solutions are} \quad y = \begin{cases} c_1 + c_2 \ln x, & \alpha = 0 \\ c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha > 0 \end{cases}$$

6. A general approach for solving ODEs with variable coefficients:

- a. Power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

but most of the ones we will see are expanded about $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

- b. Power series solutions valid only over intervals for which they converge and are analytic; they are often not valid over all of $-\infty < x < \infty$
 c. Example: $\ln(x)$ is not analytic for $x \leq 0$, so the ordinary points are $x > 0$
 d. Ordinary points: Values of x at which variable coefficients are analytic
 e. Singular points: Values of x that are not ordinary points. Test (for the 2nd-order case):

$$\text{ODE with variable coefficients: } a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

$$\text{Standard form: } y'' + P(x)y' + Q(x)y = 0,$$

$$\text{where } P(x) = \frac{a_1(x)}{a_2(x)} \quad \text{and} \quad Q(x) = \frac{a_0(x)}{a_2(x)}.$$

- f. Ordinary point at $x = x_0$ if $a_2(x_0) \neq 0$; otherwise, it's a singular point
 g. Singular points can be regular or irregular
 i. Regular if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$
 ii. Irregular if not
 h. Obtaining power series solutions about singular points requires special approaches. For regular singular points, the method of Frobenius (Sec. 5.2) is one possibility.
 i. The “famous” special functions (e.g., Bessel, Legendre, Airy, Chebyshev) are all defined by power series solutions to their associated equations.