

Lecture Outline for Friday, Oct. 16, 2026

1. A general approach for solving ODEs with variable coefficients:

a. Power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n ,$$

but most of the ones we will see are expanded about $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

- b. Power series solutions valid only over intervals for which they converge and are analytic; they are often not valid over all of $-\infty < x < \infty$
- c. Example: $\ln(x)$ is not analytic for $x \leq 0$, so solution is valid only for $x > 0$
- d. Ordinary points: Values of x at which variable coefficients are analytic
- e. Singular points: Values of x that are not ordinary points. Test (for the 2nd-order case):

$$\text{ODE with variable coefficients: } a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

$$\text{Standard form: } y'' + P(x)y' + Q(x)y = 0 ,$$

$$\text{where } P(x) = \frac{a_1(x)}{a_2(x)} \quad \text{and} \quad Q(x) = \frac{a_0(x)}{a_2(x)} .$$

- f. Ordinary point at $x = x_0$ if $a_2(x_0) \neq 0$; otherwise, it's a singular point
 - g. Singular points can be regular or irregular
 - i. Regular if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$
 - ii. Irregular if not
 - h. Obtaining power series solutions about singular points requires special approaches. For regular singular points, the method of Frobenius (Sec. 5.2) is one possibility.
 - i. The "famous" special functions (e.g., Bessel, Legendre, Airy, Chebyshev) are all defined by power series solutions to their associated equations.
2. Example of application of power series (or infinite series): Parametric Bessel equation of order ν

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2) y = 0$$

(continued on next page)

- a. The “parameter” is α
- b. Series solution (just one of the two possible solutions) is the *Bessel function of the first kind*,

$$J_\nu(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{\alpha x}{2} \right)^{2n + \nu},$$

where $\Gamma(1 + \nu)$ is the gamma function (more info in Appendix II of Zill, 6th ed.)

- c. Second series solution, which is LI from J_ν if $\nu \neq$ integer:

$$J_{-\nu}(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 - \nu + n)} \left(\frac{\alpha x}{2} \right)^{2n - \nu}$$

- d. But there are many practical applications with $\nu =$ integer. A more general second solution is the *Bessel function of the second kind*,

$$Y_\nu(\alpha x) = \frac{\cos(\nu x) J_\nu(\alpha x) - J_{-\nu}(\alpha x)}{\sin(\nu x)}.$$

Appears to be indeterminate for certain values of x when $\nu =$ integer (get 0/0).

However, it can be shown that Y_ν exists and is LI from J_ν even if ν is an integer (e.g., can use L'Hospital's rule).

See Wikipedia article “Bessel function” for plots of $J_\nu(x)$ and $Y_\nu(x)$. Note similarities to $\sin(x)$ and $\cos(x)$.

- e. Thus, the general solution of the parametric Bessel equation (for closed boundaries) is usually written

$$y(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

- f. For open boundaries, use *Hankel* functions of the first and second kind (not considered in this course):

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \quad \text{and} \quad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x) \quad i = \sqrt{-1}$$

- g. The J_ν and Y_ν functions are well known and tabulated in old references (see web link: NIST Digital Library of Mathematical Functions), but most modern math software packages have Bessel functions built in (e.g., in *Matlab*, they are `besselj` and `bessely`).
- h. See “Meet the Bessels” by Prof. Maneval for info on Bessel functions of zero order.