

Lecture Outline for Monday, Oct. 19, 2026

1. Inner products of functions and orthogonality – an important ODE/PDE solution tool

- a. Development: inner (dot) product of 3-D vectors and
- N
- dimensional vectors...

$$\mathbf{f} \cdot \mathbf{g} = \mathbf{f}^T \mathbf{g} = f_x g_x + f_y g_y + f_z g_z \quad \text{and} \quad \mathbf{f}^T \mathbf{g} = \sum_{j=1}^N f_j g_j$$

inner product of functions over the interval $[a, b]$ (can be thought of as ∞ -dimensional vectors) ...

$$\mathbf{f}^T \mathbf{g} \rightarrow \lim_{N \rightarrow \infty} \frac{b-a}{N} \sum_{j=1}^N f(x_j) g(x_j) = \int_a^b f(x) g(x) dx$$

- b. One common notation:
- $\langle f, g \rangle$
- denotes an inner product. Context does the rest.

$$\begin{aligned} \langle f, g \rangle &= \mathbf{f} \cdot \mathbf{g} && \text{3-D vectors} \\ \langle f, g \rangle &= \mathbf{f}^T \mathbf{g} && N\text{-dimensional vectors} \\ \langle f, g \rangle &= \int_a^b f(x) g(x) dx && \text{functions} \end{aligned}$$

- c. The next step: If f and g are functions and $\langle f, g \rangle = 0$, then the functions f and g are orthogonal. Yes, functions can be orthogonal too.
- d. Orthogonality makes possible the *practical* solution of many ODEs and PDEs
- e. Unique to inner products of functions – Some inner products require a weight function $w(x)$ to be meaningful. In these cases, the inner product is defined as

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx$$

For trigonometric functions (sine and cosine), $w(x) = 1$, but for Bessel functions, $w(x) = x$. We'll see why later.

2. Example: Inner products of trigonometric (circular) functions

- a. Eigenfunctions of the Fourier equation (and the wave equation PDE, as we will see soon) over a bounded interval
- $[-L, L]$
- could be

$$y_{1n}(x) = \cos\left(\frac{n\pi x}{L}\right) \quad \text{and/or} \quad y_{2n}(x) = \sin\left(\frac{n\pi x}{L}\right)$$

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- b. Consider inner product (allow eigenvalues to be different; i.e., use m and n):

$$\langle y_{1m}, y_{2n} \rangle = \int_{-L}^L y_{1m}(x) y_{2n}(x) dx = \int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

- c. Useful trigonometric identities:

$$\cos(a)\sin(b) = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$$

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

- d. inner product becomes:

$$\int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^L \sin\left[\frac{(m+n)\pi x}{L}\right] dx - \frac{1}{2} \int_{-L}^L \sin\left[\frac{(m-n)\pi x}{L}\right] dx$$

- e. Since $\sin(\alpha x)$ is an odd function, if $m \neq n$, then both integrals evaluate to zero. If $m = n$, then

$$\frac{1}{2} \int_{-L}^L \sin\left[\frac{2n\pi x}{L}\right] dx - \frac{1}{2} \int_{-L}^L \sin\left[\frac{(0)\pi x}{L}\right] dx = 0 + 0 = 0$$

- f. Since $\langle y_{1m}, y_{2n} \rangle = 0$, the functions $y_{1m}(x)$ and $y_{2n}(x)$ are orthogonal for $m \neq n$ and for $m = n$.

- g. What about the same eigenfunction with different eigenvalues ($m \neq n$)?

$$\begin{aligned} \langle y_{1m}, y_{1n} \rangle &= \int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{2} \int_{-L}^L \cos\left[\frac{(m+n)\pi x}{L}\right] dx + \frac{1}{2} \int_{-L}^L \cos\left[\frac{(m-n)\pi x}{L}\right] dx \\ &= 0 + 0 \quad (\text{orthogonal}) \end{aligned}$$

Similar result for the eigenfunction $\sin(n\pi x/L)$.

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- h. ...and the same eigenfunction with the same eigenvalues (self-product)?

$$\begin{aligned}
 \langle y_{1n}, y_{1n} \rangle &= \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^L \cos^2\left(\frac{n\pi x}{L}\right) dx \\
 &= \int_{-L}^L \left[\frac{1}{2} + \frac{1}{2} \cos\left(\frac{2n\pi x}{L}\right) \right] dx = \frac{1}{2} \int_{-L}^L dx + \frac{1}{2} \int_{-L}^L \cos\left(\frac{2n\pi x}{L}\right) dx \\
 &= L + 0 = L \quad (\text{not orthogonal; self-product is square of "length"})
 \end{aligned}$$

Similar result for the eigenfunction $\sin(n\pi x/L)$.

- i. Norm and square norm (general definitions):

$$\|y_n\| = \sqrt{\int_a^b y_n^2(x) dx} \quad \text{and} \quad \|y_n\|^2 = \int_a^b y_n^2(x) dx$$