

Lecture Outline for Wednesday, Oct. 21, 2026

1. Important general form of ODE: regular Sturm-Liouville equation

$$\frac{d}{dx} \left[r(x) \frac{dy}{dx} \right] + q(x)y + \lambda p(x)y = 0$$

subject to homogeneous boundary conditions defined over interval $[a, b]$:

$$\begin{aligned} A_1 y(a) + B_1 y'(a) &= 0, & A_1 \text{ and } B_1 \text{ not both zero} \\ A_2 y(b) + B_2 y'(b) &= 0, & A_2 \text{ and } B_2 \text{ not both zero} \end{aligned}$$

- a. some textbooks use $p(x)$ for $r(x)$ and $w(x)$ for $p(x)$
 - b. second-order with variable coefficients
 - c. form above is called “self-adjoint” form
 - d. λ is a parameter in the problem (eigenvalue)
 - e. must have $r(x), p(x) > 0$ over interval of solution
 - f. importance for our purposes:
 - i. guarantees orthogonality, completeness, representation (see supplemental reading “Sturm-Liouville Problems: Eigenfunction Orthogonality”)
 - ii. function $p(x)$ defines kernel (weight function) for inner product
2. The “Sturm-Liouville Insurance Policy” (SLIP) – proof of some of these will be shown later:
- a. There are non-trivial solutions for specific values of the parameter λ (eigenvalues).
 - b. There is an infinity of eigenvalues.
 - c. There is a smallest but not a largest eigenvalue.
 - d. The eigenvalues are real and distinct ($\lambda_1 < \lambda_2 < \lambda_3 < \dots$ such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$).
 - e. For each eigenvalue there is a single solution (eigenfunction) $y_n(x)$
 - f. The eigenfunctions corresponding to two different eigenvalues are orthogonal with respect to the weight function $p(x)$ over the interval $[a, b]$. That is,

$$\langle y_m, y_n \rangle_{p(x)} = \int_a^b y_m(x) y_n(x) p(x) dx = \begin{cases} 0, & m \neq n \\ C_m, & m = n \end{cases}$$

- g. The set of solutions to an S-L problem is complete in that the set forms a basis for the space of square-integrable functions over the interval $[a, b]$.

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x) \quad \text{with} \quad a_n = \frac{\langle f(x), y_n(x) \rangle}{\langle y_n(x), y_n(x) \rangle}$$

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In other words, just about any function over the interval $[a, b]$ can be expressed as a weighted sum of eigenfunctions.

Proof:

Multiply both sides of weighted sum expression by $y_m(x)$ and evaluate the inner product. Don't forget the weight function $p(x)$:

$$\int_a^b f(x) y_m(x) p(x) dx = \sum_{n=1}^{\infty} a_n \int_a^b y_m(x) y_n(x) p(x) dx$$

Because of the SLIP, we know that the eigenfunctions $y_n(x)$ are orthogonal. Thus,

$$\int_a^b y_m(x) y_n(x) p(x) dx = 0 \text{ for } m \neq n,$$

so

$$\int_a^b f(x) y_m(x) p(x) dx = a_m \int_a^b y_m(x) y_m(x) p(x) dx$$

Expressed in inner product notation,

$$\langle f(x), y_m(x) \rangle = a_m \langle y_m(x), y_m(x) \rangle \rightarrow a_m = \frac{\langle f(x), y_m(x) \rangle}{\langle y_m(x), y_m(x) \rangle}$$

The expression for a_n on the previous page follows after changing the index from m to n .

3. The SLIP is such a valuable set of properties that it is worth determining whether a given problem is a Sturm-Liouville problem. To find out, convert it to *self-adjoint form* shown at the top of the previous page. A second-order ODE of the form

$$a(x) y'' + b(x) y' + c(x) y + \lambda d(x) y = 0$$

can be converted to the equivalent self-adjoint form

$$\frac{d}{dx} \left[r(x) \frac{dy}{dx} \right] + q(x) y + \lambda p(x) y = 0$$

if $a(x) \neq 0$ everywhere.

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4. Conversion to self-adjoint form:

- a. Compute the integrating factor $\mu(x)$ (watch out for $a(x) = 0$ for any x over the bounded interval):

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right)$$

- b. Compute the variable coefficients of the S-L adjoint form:

$$r(x) = \mu(x) \quad q(x) = \frac{c(x)}{a(x)} \mu(x) \quad p(x) = \frac{d(x)}{a(x)} \mu(x)$$

- c. Verify that $r(x), p(x) > 0$ over the applicable interval of the solution.
- d. Note that $p(x)$ serves as the weighting function in the inner product that tests orthogonality.

5. Example #1: Are $y'' + \lambda y = 0$ and $y'' - \lambda y = 0$ S-L equations?

- a. First note that $a(x) = 1$, $b(x) = 0$, $c(x) = 0$, and $d(x) = \pm 1$. Then:

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp(0) = 1$$

$$\rightarrow r(x) = \mu(x) = 1 \quad q(x) = \frac{c(x)}{a(x)} \mu(x) = 0 \quad p(x) = \frac{d(x)}{a(x)} \mu(x) = \pm 1$$

- b. Self-adjoint form of Fourier and modified Fourier equations:

$$\frac{d}{dx} \left[\frac{dy}{dx} \right] \pm \lambda y = 0 \quad \text{or} \quad y'' \pm \lambda y = 0, \text{ of course}$$

- c. Note that $r(x) > 0$ for both equations but that $p(x) > 0$ for the Fourier equation and $p(x) < 0$ for the modified Fourier equation. Thus, the Fourier equation satisfies the SLIP and is an S-L equation but not the modified Fourier equation. Also note that $p(x) = 1$ for the Fourier equation.