ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Monday, Sept. 21, 2026

1. Orthogonal matrices ($A^{-1} = A^T$, which implies that $A^T A = I$)

   a. A is orthogonal iff its columns form an orthonormal set (orthonormal is orthogonal with each vector having a length of 1; i.e., $|\mathbf{x}| = \mathbf{x}^T\mathbf{x} = 1$)
   b. Orthogonal matrices are not usually symmetric (The only orthogonal *and* symmetric matrix is I because such a matrix satisfies $A^{-1} = A^T = A$.)
   c. Important application: Various kinds of factorizations (also called decompositions) and diagonalizations, which can improve the efficiency of difficult computations and reveal conditioning (sensitivity to round-off errors) of matrices
   d. From MathWorks Help Center: "Orthogonal…matrices are desirable for numerical computation because they preserve length, preserve angles, and do not magnify errors."
   e. If A is orthogonal, then $\det(A) = \pm 1$
   f. If A has complex elements, then A is called *unitary* if $A^{-1} = A^*$, where * indicates the Hermitian, or complex conjugate, transpose (H is sometimes used instead of *.)
   g. Example: Use *Matlab* to check that $A^{-1} = A^T$ and that the columns form an orthonormal set of vectors. Also verify that $\det(A) = \pm 1$.

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

2. Diagonalization

   a. Recall that an eigensystem can be summarized compactly as

$$A\mathbf{x}_i = \lambda\mathbf{x}_i \quad i = 1, 2, 3, \ldots, N \quad \rightarrow \quad AX = X\Lambda \text{, where}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

   If the eigenvectors are linearly independent (LI), then X is nonsingular, and $A = X\Lambda X^{-1}$.

(*continued on next page*)

b. In general, an $N \times N$ matrix A can be sometimes be expressed as $A = PDP^{-1}$, where D is a diagonal matrix; this is called *diagonalization*.
c. A does not have to be symmetric or orthogonal to be diagonalizable.
d. Theorem: An $N \times N$ matrix A is diagonalizable iff A has N LI eigenvectors.
e. Theorem: If $N \times N$ matrix A has N LI eigenvectors, then it is diagonalizable.
f. Theorem: If $N \times N$ matrix A has N distinct eigenvalues, then it is guaranteed to be diagonalizable because such a matrix is guaranteed to have LI eigenvectors (but it might be diagonalizable even if the eigenvalues are not distinct, that is, if some are repeated, as long as its eigenvectors are LI. Example: the identity matrix).
g. Theorem: An $N \times N$ matrix A can be orthogonally diagonalized iff A is symmetric. (*Orthogonal diagonalization* means that P is orthogonal.) Converse is also true: All symmetric matrices are orthogonally diagonalizable.
h. Many ways to diagonalize a matrix.
i. Example: Attempt to diagonalize the following matrix using its eigensystem:

$$A = \begin{bmatrix} -5 & 9 \\ -6 & 10 \end{bmatrix}$$

Eigenvalues are 1 and 4. Determination of eigenvectors:

$$\left(A - \lambda_1 I\right)\mathbf{x}_1 = \mathbf{0} \quad \rightarrow \quad \left[\begin{array}{cc|c} -5-1 & 9 & 0 \\ -6 & 10-1 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -6 & 9 & 0 \\ -6 & 9 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -6 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \quad -6x_{11} + 9x_{12} = 0 \quad \rightarrow \quad x_{12} = \frac{2}{3}x_{11} \quad \rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\left(A - \lambda_2 I\right)\mathbf{x}_2 = \mathbf{0} \quad \rightarrow \quad \left[\begin{array}{cc|c} -5-4 & 9 & 0 \\ -6 & 10-4 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -9 & 9 & 0 \\ -6 & 6 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -9 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \quad -9x_{21} + 9x_{22} = 0 \quad \rightarrow \quad x_{22} = x_{21} \quad \rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus,

$$A = X\Lambda X^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

If the eigenvectors are normalized (to allow checking with *Matlab*),

$$A = X\Lambda X^{-1} = \begin{bmatrix} 0.8321 & 0.7071 \\ 0.5547 & 0.7071 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0.8321 & 0.7071 \\ 0.5547 & 0.7071 \end{bmatrix}^{-1}$$

Note that X is not orthogonal because A is not symmetric.