

Lecture Outline for Friday, Oct. 23, 2026

1. Another example of conversion to Sturm-Liouville equation in self-adjoint form: Parametric Bessel's equation

$$x^2 y'' + xy' + (\lambda x^2 - \nu^2) y = 0$$

Identify the variable coefficients:

$$a(x) = x^2 \qquad b(x) = x \qquad c(x) = -\nu^2 \qquad d(x) = x^2$$

Compute the integrating factor and other S-L equation factors:

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp\left(\int \frac{x}{x^2} dx\right) = \exp\left(\int \frac{1}{x} dx\right) = e^{\ln x} = x$$

$$\rightarrow \quad r(x) = \mu(x) = x \qquad q(x) = \frac{-\nu^2}{x^2} x = \frac{-\nu^2}{x} \qquad p(x) = \frac{x^2}{x^2} x = x.$$

Because of the requirement that $r(x) > 0$ and $p(x) > 0$, this solution seems to apply only for $x > 0$; however, we will soon see that it is also valid at $x = 0$.

Self-adjoint form of Bessel's equation:

$$\frac{d}{dx} \left[x \frac{dy}{dx} \right] - \frac{\nu^2}{x} y + \lambda xy = 0$$

Significance: We now know the kernel $p(x)$ used in the inner product: $p(x) = x$.

2. Example of applying self-adjoint form: Solve the BVP

$$x^2 y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

- a. Compare to general form of the Bessel equation $x^2 y'' + xy' + (\lambda x^2 - \nu^2) y = 0 \rightarrow \nu = 0$
- b. Potential general solution form for closed boundaries:

$$y(x) = c_1 J_0(\sqrt{\lambda} x) + c_2 Y_0(\sqrt{\lambda} x) \rightarrow y'(x) = -c_1 \sqrt{\lambda} J_1(\sqrt{\lambda} x) - c_2 \sqrt{\lambda} Y_1(\sqrt{\lambda} x)$$

(continued on next page)

See Eqn. (22) in Sec. 5.3.1 of the textbook:

$$\begin{aligned}\frac{d}{dx}[x^{-\nu}J_{\nu}(x)] &= -x^{-\nu}J_{\nu+1}(x) \rightarrow \frac{d}{dx}[x^{-0}J_0(x)] = -x^{-0}J_1(x) = -J_1(x) \\ &\rightarrow \frac{d}{dx}[J_0(x)] = -J_1(x)\end{aligned}$$

Also applies to Bessel function of the second kind, so

$$\frac{d}{dx}[Y_0(x)] = -Y_1(x)$$

Apply BC #1:

$$y'(0) = 0 = -c_1\sqrt{\lambda}J_1(0) - c_2\sqrt{\lambda}Y_1(0) = -c_1\sqrt{\lambda}(0) - c_2\sqrt{\lambda}(-\infty)$$

Since $Y_1(0) \rightarrow -\infty$ (as does $Y_0(0)$), $Y_0(\sqrt{\lambda}x)$ is not a viable solution.

Apply BC #2:

$$y(1) = 0 = c_1J_0(\sqrt{\lambda})$$

This implies that λ can only have values for which $\sqrt{\lambda_n} = r_n$, $n = 1, 2, 3, \dots$, where r_n are the roots (zeros) of J_0 .

- c. First four roots of J_0 : 2.4048, 5.5201, 8.6537, 11.7915 (see Table 5.3.1 in the textbook)
- d. Try evaluating inner product with and without $p(x) = x$; this is the focus of Lab #4:

$$\left\langle J_0(\sqrt{\lambda_m}x), J_0(\sqrt{\lambda_n}x) \right\rangle = \int_0^1 J_0(\sqrt{\lambda_m}x) J_0(\sqrt{\lambda_n}x) dx = ?$$

or

$$\left\langle J_0(\sqrt{\lambda_m}x), J_0(\sqrt{\lambda_n}x) \right\rangle = \int_0^1 x J_0(\sqrt{\lambda_m}x) J_0(\sqrt{\lambda_n}x) dx = ?$$

Notice the factor x at the beginning of the second integrand.

- e. An important question, especially as we move into partial differential equations: Under what conditions are the Bessel function solutions orthogonal?

(continued on next page)

3. Orthogonality conditions on solutions to Sturm-Liouville problems

- a. Consider the Sturm-Liouville equation in self-adjoint form for two different eigenvalues

$$\frac{d}{dx} \left[r(x) \frac{dy_m}{dx} \right] + q(x) y_m + \lambda_m p(x) y_m = 0$$

$$\frac{d}{dx} \left[r(x) \frac{dy_n}{dx} \right] + q(x) y_n + \lambda_n p(x) y_n = 0$$

- b. Multiplying the first equation by y_n and the second by y_m , subtracting the two equations, and finally integrating by parts from $x = a$ to $x = b$ yields

$$(\lambda_m - \lambda_n) \int_a^b p(x) y_m(x) y_n(x) dx = r(b) [y_m(b) y'_n(b) - y_n(b) y'_m(b)] - r(a) [y_m(a) y'_n(a) - y_n(a) y'_m(a)]$$

- c. Note that the left-hand side includes the inner product. One implication of this result is that if $r(x) > 0$ everywhere, then the boundary conditions at $x = a$ and $x = b$ must be homogenous if the solutions y_m and y_n are to be orthogonal. If $m \neq n$ and the BCs are homogeneous, then the right-hand side equals zero (proof later).

$$\text{Homogenous BCs at } x = a \rightarrow y_m(a) y'_n(a) - y_n(a) y'_m(a) = 0$$

$$\text{Homogenous BCs at } x = b \rightarrow y_m(b) y'_n(b) - y_n(b) y'_m(b) = 0$$

- d. Another implication is that y_m and y_n can be orthogonal if $r(x) = 0$ at one of the boundaries and the BC at the other boundary is homogeneous.