ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Monday, Oct. 26, 2026

1. Orthogonality conditions on solutions to Sturm-Liouville problem

   a. Consider the Sturm-Liouville equation in self-adjoint form for two different eigenvalues

$$\frac{d}{dx}\left[r(x)\frac{dy_m}{dx}\right]+q(x)y_m+\lambda_m p(x)y_m=0$$

$$\frac{d}{dx}\left[r(x)\frac{dy_n}{dx}\right]+q(x)y_n+\lambda_n p(x)y_n=0$$

   b. Multiplying the first equation by y_n and the second by y_m, subtracting the two equations, and finally integrating by parts from $x = a$ to $x = b$ yields

$$(\lambda_m-\lambda_n)\int_a^b p(x)y_m(x)y_n(x)\,dx = r(b)\left[y_m(b)y_n'(b)-y_n(b)y_m'(b)\right] - r(a)\left[y_m(a)y_n'(a)-y_n(a)y_m'(a)\right]$$

   c. Note that the left-hand side includes the inner product. One implication of this result is that if $r(x) > 0$ everywhere, then the boundary conditions at $x = a$ and $x = b$ must be homogenous if the solutions y_m and y_n are to be orthogonal. If $m \neq n$ and the BCs are homogeneous, then the right-hand side equals zero. (See below.)

Homogenous BCs at $x = a$ $\rightarrow$ $y_m(a)y_n'(a)-y_n(a)y_m'(a)=0$

Homogenous BCs at $x = b$ $\rightarrow$ $y_m(b)y_n'(b)-y_n(b)y_m'(b)=0$

   d. Another implication is that y_m and y_n can be orthogonal if $r(x) = 0$ at one of the boundaries and the BC at the other boundary is homogeneous.

2. Singular Sturm-Liouville problem

   a. Addresses cases when $r(x) > 0$ is not satisfied at one or both boundaries

   b. Right-hand side of equation in item 1b above is zero when:

      i. $r(a) = 0$ and $y_m(b)y_n'(b)-y_n(b)y_m'(b)=0$
      ii. $r(b) = 0$ and $y_m(a)y_n'(a)-y_n(a)y_m'(a)=0$
      iii. $r(a) = r(b) = 0$ and no BCs are specified at $x = a$ or $x = b$
      iv. $r(a) = r(b)$ and the BCs are $y(a) = y(b)$ and $y'(a) = y'(b)$ (periodic BCs)

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c. Warning: The solutions $\{y_n\}$ are orthogonal if $r(a) = 0$ and/or $r(b) = 0$ provided that $y_m(x)$ and $y_n(x)$ are bounded (i.e., do not go to $\pm\infty$) at the corresponding boundary.

3. Example: Recall the parametric Bessel equation

$$x^2y'' + xy' + \left(\lambda x^2 - \nu^2\right)y = 0$$

Conversion to Sturm-Liouville equation in self-adjoint form yields $r(x) = x$, so $r(0) = 0$. We considered the BVP

$$x^2y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

General solution is

$$y(x) = c_1 J_0\left(\sqrt{\lambda}x\right) + c_2 Y_0\left(\sqrt{\lambda}x\right),$$

but this is a singular S-L problem because $r(0) = 0$. Also, because $Y_0(0) \to -\infty$, $Y_0\left(\sqrt{\lambda}x\right)$ is not a viable solution (it's not bounded). However, we can show that

$$y_m(1)y_n'(1) - y_n(1)y_m'(1) = (0)y_n'(1) - (0)y_m'(1) = 0$$

because the second BC $y(1) = 0$ applies to all solutions. Thus, there are nontrivial, orthogonal solutions to this BVP.

4. Note that, if the BCs at $x = a$ are homogeneous, then

$$\begin{aligned} A_1y_m(a) + B_1y_m'(a) &= 0 \\ A_1y_n(a) + B_1y_n'(a) &= 0 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Can only be true for nonzero coefficient vector $[A_1\ B_1]^T$ if determinant of matrix is zero, so

$$\begin{vmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{vmatrix} = 0 \quad \rightarrow \quad y_m(a)y_n'(a) - y_n(a)y_m'(a) = 0$$

Also satisfied if either $A_1 = 0$ or $B_1 = 0$. For example, if $A_1 \neq 0$ and $B_1 = 0$, then $y_m(a) = 0$ and $y_n(a) = 0$, which still guarantees that $y_m(a)y_n'(a) - y_n(a)y_m'(a) = 0$.

Similar results for other general BC at $x = b$. Thus, the right-hand side of the equation in 1b above equals zero even if $r(a) \neq 0$ and $r(b) \neq 0$ and therefore the inner product of the eigenfunctions is equal to zero; i.e., the eigenfunctions are orthogonal.

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5. New major topic: Partial differential equations (PDEs). Applications:

    a. Heat transfer
    b. Diffusion
    c. Wave propagation (acoustic, mechanical, electromagnetic, etc.)
    d. Acoustics
    e. Highway traffic
    f. Many, many others

6. General form and classification (significance of classification most apparent in Chap. 16):

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

    a. Hyperbolic: $B^2 - 4AC > 0$
    b. Parabolic: $B^2 - 4AC = 0$
    c. Elliptic: $B^2 - 4AC < 0$