ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Wednesday, Oct. 28, 2026

1. Types of PDEs and conditions on which we will concentrate

   a. 1-D heat equation: $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}, \quad k\ge 0$ (parabolic)

   b. 2-D wave equation: $a^2\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$ (hyperbolic)

   c. 2-D Laplace's equation: $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ (elliptic)

   d. All three PDEs above can be extended to more physical dimensions
   e. All problems have boundary conditions and initial conditions
   f. Many are eigenvalue problems
   g. Later: Numerical solutions to PDEs

2. Method of separation of variables (SOV)

   a. Overall solution assumed to be product of functions in each independent variable.
   b. Heat equation example:

   $$\text{Let } u(x,t)=X(x)T(t)$$

   Substitution into 1-D heat equation (in 1a above) leads to

   $$k\frac{\partial^2(XT)}{\partial x^2}=\frac{\partial(XT)}{\partial t} \quad\rightarrow\quad kX''T=XT' \quad\rightarrow\quad \frac{X''}{X}=\frac{T'}{kT},$$

   where constant k is the thermal diffusivity. The single and double primes indicate derivatives with respect to the appropriate independent variable (x in X and t in T). The goal is to make one side dependent on only one variable. In this case, the left-hand side depends only on x and the right-hand side only on t.

   c. Introduce a new quantity, the separation constant:

   $$\frac{X''}{X}=\frac{T'}{kT}=-\lambda$$

   One view of why this works: Consider a third arbitrary independent variable; call it z.

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Take the derivative of the separated equation with respect to z:

$$\frac{\partial}{\partial z}\left(\frac{X''}{X}\right)=\frac{\partial}{\partial z}\left(\frac{T'}{kT}\right)=0$$

Because both derivatives are zero, each side of the separated equation must be equal to a constant. Because the undifferentiated functions are equal, they must equal the same constant.

d. Result is two linked ODEs:

$$\frac{X''}{X}=-\lambda \quad \rightarrow \quad X''+\lambda X=0 \quad \text{and} \quad \frac{T'}{kT}=-\lambda \quad \rightarrow \quad T'+\lambda kT=0$$

e. The solution to the ODE in one independent variable could be an infinite sum of eigenfunctions corresponding to different eigenvalues (superposition principle).

f. Two questions to be answered soon:

   i. Which ODE should be associated with the constants in the original PDE (e.g., the constant *k* in the heat equation example above)?
   ii. Why do we use $-\lambda$ instead of λ for the separation constant?

g. It is not always possible to find a solution with this method; some PDE solutions are not separable.

3. Example application of separation of variables method: Heat equation

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \quad \text{with BCs } u(0,t)=0 \text{ and } u(L,t)=0, \text{ and IC } u(x,0)=f(x)$$

"BC" = boundary condition; "IC" = initial condition

a. As shown above, application of SOV method results in two linked ODEs:

$$X''+\lambda X=0 \quad \text{and} \quad T'+\lambda kT=0$$

b. Boundary conditions become

$$u(0,t)=X(0)T(t)=0 \quad \rightarrow \quad X(0)=0$$
$$u(L,t)=X(L)T(t)=0 \quad \rightarrow \quad X(L)=0$$

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c. The $X(x)$ problem is an S-L problem, and the $T(t)$ problem is an IVP with a first-order ODE:

$$X'' + \lambda X = 0 \quad \text{with} \quad X(0) = 0 \text{ and } X(L) = 0$$

$$T' + \lambda kT = 0 \quad \rightarrow \quad T(t) = c_3 e^{-k\lambda t}$$

d. Since the X problem has closed boundaries and is an S-L problem (homogeneous BCs), the most convenient solution form is

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

e. Apply BCs to obtain eigenfunctions and eigenvalues

$$X_n(x) = c_{2n} \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Do not include the $n = 0$ case since that leads to the trivial solution.

f. Each eigensolution to the full problem (x and t domains) has the form:

$$u_n(x,t) = X(x)T(t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$$

The unknown coefficients in the $X(x)$ and $T(t)$ solutions have been subsumed into the composite constant A_n; that is, $A_n = c_{2n}\, c_3$.

g. Each eigensolution u_n is a solution, but a linear combination of eigensolutions is also a solution. Full solution to the PDE for the general case:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$$