

Lecture Outline for Monday, Sept. 28, 2026

1. Implications and properties of SVD and some ways to find U , Σ , and V

- a. $A = U\Sigma V^T = \sum_{j=1}^N \sigma_j \mathbf{u}_j \mathbf{v}_j^T$, where $\mathbf{u}_j \mathbf{v}_j^T$ is an outer product, each of which is $M \times N$.
- i. The outer products have rank = 1. (See similar HW #2 problem.)
 - ii. A is a weighted sum of rank-1 $M \times N$ matrices.
 - iii. The weights (σ_j) grow progressively smaller.
 - iv. If any of the singular values are zero or too small to matter (lost in the noise, for example), then A can be represented by subsets of the orthogonal vectors $\{\mathbf{u}_i\}_{i=1 \text{ to } r}$ and $\{\mathbf{v}_i\}_{i=1 \text{ to } r}$, where $r < N$.
 - v. The summation terms for zero or nearly zero singular values can be ignored. (Example given in image processing demo later.)
- b. Use complex conjugate transpose for complex matrix A .
- c. U and V are both orthogonal ($U^{-1} = U^T$ and $V^{-1} = V^T$); thus,
 $A = U\Sigma V^T \rightarrow AV = U\Sigma \rightarrow A\mathbf{v}_i = \sigma_i \mathbf{u}_i$
- d. $A = U\Sigma V^T \rightarrow A^T = V\Sigma^T U^T \rightarrow A^T U = V\Sigma \rightarrow A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i$
- e. $A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T$ (an $N \times N$ system)
 $\rightarrow (A^T A)V = V\Sigma^T \Sigma \rightarrow (A^T A)\mathbf{v}_i = \sigma_i^2 \mathbf{v}_i$ for $i = 1$ to N
 σ_i^2 are the eigenvalues of $A^T A$, and $\{\mathbf{v}_i\}_{i=1 \text{ to } N}$ are the eigenvectors.
If $N > M$, then at most only M of the N eigenvalues will be nonzero.
- f. $AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma \Sigma^T U^T$ (an $M \times M$ system)
 $\rightarrow (AA^T)U = U\Sigma \Sigma^T \rightarrow (AA^T)\mathbf{u}_i = \sigma_i^2 \mathbf{u}_i$ for $i = 1$ to M
 σ_i^2 are also the eigenvalues of AA^T , and $\{\mathbf{u}_i\}_{i=1 \text{ to } M}$ are the eigenvectors
If $M > N$, then at most only N of the M eigenvalues will be nonzero.
- g. If A is symmetric, then $A^T A = AA^T = A^2$, so $\lambda_i^2 = \sigma_i^2 \rightarrow |\lambda_i| = |\sigma_i|$ (sign ambiguity);
 $\{\mathbf{u}_i\}$ and $\{\mathbf{v}_i\}$ are the same and are the eigenvectors of A
- h. If A is not symmetric, then neither $\{\mathbf{u}_i\}$ nor $\{\mathbf{v}_i\}$ are the eigenvectors of A

2. Applications and examples

- a. To solve a system $A\mathbf{x} = \mathbf{b}$ (this is for overdetermined and square systems; underdetermined requires interpretation):

$$U\Sigma V^T \mathbf{x} = \mathbf{b} \rightarrow \Sigma V^T \mathbf{x} = U^T \mathbf{b} \rightarrow V^T \mathbf{x} = \Sigma^{-1} U^T \mathbf{b} \rightarrow \mathbf{x} = V \Sigma^{-1} U^T \mathbf{b}$$

(continued on next page)

Since Σ is diagonal (in the “economy-sized” decomposition), its inverse is

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/\sigma_N \end{bmatrix}_{N \times N}.$$

Matrix expression for $\mathbf{x}$ is equivalent to

$$\mathbf{x} = \sum_{j=1}^N \left(\frac{\mathbf{u}_j^T \mathbf{b}}{\sigma_j} \right) \mathbf{v}_j$$

In ill-conditioned systems, some $\mathbf{u}$ vectors are nearly orthogonal to $\mathbf{b}$. (In singular systems, some *are* orthogonal.) The corresponding singular values are very small, so those terms “corrupt” the solution $\mathbf{x}$. Workaround: Omit the problematic terms with small σ_j .

- b. Example #1: Compare eigenvalues to singular values of the symmetric matrix A (use *Matlab* `eig` and `svd` commands):

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ 4 & 5 & 7 \end{bmatrix}$$

- i. Notice order of singular values
- ii. Check orthogonality of columns of U and V
- iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
- iv. What is the rank of this matrix?
- v. What are the ranks of $\mathbf{u}_1 \mathbf{v}_1^T$, $\mathbf{u}_2 \mathbf{v}_2^T$, and $\mathbf{u}_3 \mathbf{v}_3^T$?

- c. Example #2: Compare eigenvalues to singular values of the singular matrix A :

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ 4 & 1 & -6 \end{bmatrix}$$

- i. Notice order of singular values
- ii. Check orthogonality of columns of U and V
- iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
- iv. What is the rank of this matrix?
- v. What are the ranks of $\mathbf{u}_1 \mathbf{v}_1^T$, $\mathbf{u}_2 \mathbf{v}_2^T$, and $\mathbf{u}_3 \mathbf{v}_3^T$?

- d. Example #3: Image processing/storage demonstration.