

Lecture Outline for Friday, Oct. 2, 2026

1. New Topic: Ordinary differential equations (ODEs) – scope and coverage

- a. 2nd order linear equations are some of the most important equations of mathematical physics
- b. Important special cases will be our focus
- c. Linear N^{th} order differential equation – general form:

$$a_n(x) \frac{d^n y}{dx^n}(x) + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x) y = g(x)$$

- i. Coefficients can be constants (not functions of x) or variables (functions of x)
 - ii. Homogeneous if $g(x) = 0$; otherwise, nonhomogeneous
- d. Initial value problems (IVPs) have initial conditions and unique solutions:

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots \quad y^{(n-1)}(x_0) = y_{n-1} \quad (\text{defined at } x = x_0)$$

- e. Boundary value problems (BVPs):
 - i. Boundary conditions: Dependent variable and/or its derivatives defined at two or more points
 - ii. Can have no solution, a unique solution, or many solutions
 - iii. Special class of BVPs: eigenvalue problems (EVPs)
 - iv. EVPs will be the main emphasis for this part of the course

2. Solvable BVP class #1: Constant coefficients

- a. Chapter 2 covers 1st order with constant coefficients (skim)
- b. Section 3.3 covers 2nd order with constant coefficients
- c. Proposed solution and consequence
- d. Cases and solution forms (special cases: Fourier equation $y'' + a^2 y = 0$ and modified Fourier equation $y'' - a^2 y = 0$)

3. Solvable BVP class #2: Variable coefficients

- a. Some general forms considered (not an exhaustive list); primary focus is on first two, which are applicable to common PDEs:

(continued on next page)

- i. Cauchy-Euler equation: $ax^2y'' + bxy' + cy = 0$
- ii. Bessel equation: $x^2y'' + xy' + (x^2 - n^2)y = 0$ (also parametric and modified)
- iii. Legendre equation: $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$
- iv. Airy equation: $y'' \pm a^2xy = 0$
- v. Chebyshev equation: $(1 - x^2)y'' - xy' + a^2y = 0$

- b. Proposed solution and consequence
- c. Cases and solution forms (special functions)

4. Solution strategies

- a. Identify class of ODE
- b. Applicable solution structure (What should solution look like?)
- c. Corresponding solution method
- d. Starting point: homogenous solutions

5. Important theorems and concepts applicable to ODEs

- a. IVPs have unique solutions (not true for BVPs)
- b. Superposition principle; a linear combination of solutions to a homogeneous ODE over an interval is also a solution over the same interval
- c. Corollaries: 1) A constant multiple of a solution is also a solution; 2) homogeneous ODEs always possess at least the trivial solution $y = 0$
- d. Linearly independent vs. linearly dependent solutions (analogy to vectors)
- e. An N^{th} order homogeneous linear ODE has a fundamental set of N linear independent solutions. The general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

- f. Nonhomogeneous ODEs: general solution = complementary solution + particular solution (complementary solution is the full solution set of the corresponding homogenous ODE)
- g. Superposition also applies to particular solutions: If y_{p1} is a solution of the DE with $g_1(x)$, y_{p2} is a solution with $g_2(x)$, etc., then $y_{p1} + y_{p2} + \dots$ is a solution to

$$a_n(x)\frac{d^ny}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots$$

6. Solution of nonhomogeneous linear ODEs with constant coefficients (not emphasized here)

- a. You can guess or...
- b. Method of undetermined coefficients (Sec. 3.4 of Zill, 6th ed.) – doesn't work for all forcing functions
- c. Method of variation of parameters (Sec. 3.5 of Zill, 6th ed.) – more general & complicated
- d. Use the annihilator method (see web link) – annihilators do not always exist