ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Friday, Oct. 30, 2026

1. Example application of separation of variables method: Heat equation

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\quad \text{with}\quad u(0,t)=0,\quad u(L,t)=0,\quad \text{and}\quad u(x,0)=f(x)$$

   a. Full solution of the PDE for the general case:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)e^{-kn^2\pi^2 t/L^2}$$

   b. Determination of coefficients in summation:

   Apply IC: $f(x)=u(x,0)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$

   Multiply both sides by the *m*th eigenfunction and the weighting function [$p(x) = 1$ in this case] and integrate over interval of interest:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx=\sum_{n=1}^{\infty}A_n\int_0^L\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

   Because the $X(x)$ problem is an S-L problem, the eigenfunctions are orthogonal, and the inner products for $m \neq n$ are therefore zero. Thus,

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx=A_m\int_0^L\sin^2\left(\frac{m\pi x}{L}\right)dx=A_m\int_0^L\left[\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2m\pi x}{L}\right)\right]dx=A_m\left(\frac{L}{2}\right)$$

$$\rightarrow\quad A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

   For some initial value functions $f(x)$, the integral could be evaluated analytically. In general or perhaps to save time, the integral could be evaluated numerically.

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2. Interpretation of solution

$$u(x,t)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)e^{-kn^2\pi^2t/L^2}$$

   a. How do we break it down?
   b. Does it satisfy BCs?
   c. What can we learn from it?
   d. Does it make sense? Behavior as $n \to \infty$ and $t \to \infty$
   e. Behavior of coefficients in infinite series as $n \to \infty$:

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

      i. As $n \to \infty$, the eigenfunction $\sin(n\pi x/L)$ consists of a large number of cycles that span $x = 0$ to $x = L$.
      ii. If $f(x)$ is relatively smooth, then it is almost constant over each individual cycle of $\sin(n\pi x/L)$.
      iii. The integral from 0 to L can be decomposed into a sum of n different integrals over the individual cycles of $\sin(n\pi x/L)$.
      iv. Each of those integrals evaluates to almost zero because there are almost equal areas under the curve above and below the x-axis. Thus, $A_n \to 0$ as $n \to \infty$.

   f. *Matlab* simulation

3. Not always possible to find a solution using the SoV method; some PDE solutions are not separable. Which of the following PDEs can be solved via SOV, and which cannot?

   a. $x\dfrac{\partial u}{\partial x}=y\dfrac{\partial u}{\partial y}$

   b. $a^2\dfrac{\partial^2 u}{\partial x^2}-g=\dfrac{\partial^2 u}{\partial t^2}$, where g is a constant

   c. $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial y}+u$

   d. $a^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=\dfrac{\partial u}{\partial t}$