

Lecture Outline for Monday, Aug. 31

1. Matrix equation from nonorthogonal unit vector example:

$$\begin{bmatrix} 8 & 40 \\ 40 & 200 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 11.24 \\ 56.20 \end{bmatrix} \quad (\text{note that } 5 \times 11.24 = 56.20)$$

Does this system of equations have a solution? If so, how many?

Is the matrix singular or nonsingular? Does the matrix have an inverse? What is the determinant of the matrix?

2. Solution of $A\mathbf{x} = \mathbf{b}$ using the inverse: Start with the 1×1 (scalar) analogy: $ax = b$
 - a. $1/a$ and a^{-1} : same thing but different perspective
 - b. Key requirement of a^{-1} for solution
 - c. How a^{-1} is used to get solution
 - d. Roles of a and b in determining the number and nature of solution(s)
3. Solution for the $N \times N$ case: $A\mathbf{x} = \mathbf{b}$
 - a. Inverse (“reciprocal”) matrix A^{-1}
 - b. Key requirement of A^{-1} for solution
 - c. How A^{-1} is used to get solution
 - d. How to find A^{-1} ?
4. For an $N \times N$ (square) system, the following statements are equivalent for the purpose of determining the solvability of the problem.
 - a. $A\mathbf{x} = \mathbf{b}$ has a unique solution
 - b. A has a unique inverse (A^{-1})
 - c. A is nonsingular
 - d. $\det(A) = |A| \neq 0$
 - e. A has full rank (i.e., $\text{rank}(A) = N$) – more on this later

5. Route to finding solutions of $A\mathbf{x} = \mathbf{b}$ (implicit inverse computation)

- Form augmented matrix $[A \mid \mathbf{b}]$
- Process: for augmented matrix, reduce (transform) a system to an easier-to-solve form
- $A\mathbf{x} = \mathbf{b}$ becomes $U\mathbf{x} = \mathbf{d}$ and solution ensues (U is upper triangular)
- Method: row reduction using elementary row operations (EROs); Gaussian elimination (stops at REF) or Gauss-Jordan elimination (stops at RREF)
 - Multiply a row (j) by a value (c)
 - Add scaled row to another row (k) to eliminate leading zero
 - Interchange rows j and k if necessary

Example Problems in Solving Systems of Linear Equations

Prob. 1:

$$\begin{array}{l} 3x_1 - x_2 + x_3 = -1 \\ 9x_1 - 2x_2 + x_3 = -9 \\ 3x_1 + x_2 - 2x_3 = -9 \end{array} \quad \text{Augmented matrix:} \quad \left[\begin{array}{ccc|c} 3 & -1 & 1 & -1 \\ 9 & -2 & 1 & -9 \\ 3 & 1 & -2 & -9 \end{array} \right]$$

Probs. 2a and 2b:

$$\begin{array}{l} x + 2y - 5z = 2 \\ 2x - 3y + 4z = 4 \\ 4x + y - 6z = 8 \end{array} \quad \text{Augmented matrix:} \quad \left[\begin{array}{ccc|c} 1 & 2 & -5 & 2 \\ 2 & -3 & 4 & 4 \\ 4 & 1 & -6 & 8 \end{array} \right] \quad \text{What if } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} ?$$

6. Going further...

- Upper triangular (matrix entries below main diagonal are zero)
- Row-echelon form (REF; upper triangular with 1s on main diagonal)
- Reduced row-echelon form (`rref` in *Matlab*; EROs until identity matrix is obtained or can't go any further)

7. Singular systems: rank and consistency

- Consistent: has at least one solution or infinitely many solutions (other numbers of solutions – e.g., 3 – are not possible). Example in 2-D: Solution space is a point or a line
- Inconsistent: No solutions: Example in 2-D: Parallel lines – no intersecting point
- Rank: Max. no. of independent row vectors in a matrix (can use `rank` command in *Matlab*) – examples coming soon
- Relationship between rank and row-echelon (or reduced row-echelon) form