ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Wednesday, Nov. 4, 2026

1. Continue wave equation example

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad v > 0, \quad 0 \le x \le L, \quad t \ge 0$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

   a. The BCs $u(0, t) = 0 \rightarrow X(x) = 0$ and $u(L, t) = 0 \rightarrow X(L) = 0$ lead to the orthogonal spatial eigenfunctions

$$X_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

   b. Each eigensolution to the full PDE with $A_n = c_2c_4$ and $B_n = c_2c_5$ has the form:

$$u_n(x,t) = X_n(x)T_n(t) = \left[A_n \cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi vt}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

   c. Full solution to the PDE for the general case (linear combination of eigensolutions):

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi vt}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

   d. First time derivative (needed when second initial condition is applied):

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[-A_n \frac{n\pi v}{L} \sin\left(\frac{n\pi vt}{L}\right) + B_n \frac{n\pi v}{L} \cos\left(\frac{n\pi vt}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

   e. Determination of coefficients in summation. Two sets of coefficients determined by two initial conditions (ICs):

Apply IC #1: $u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

(*continued on next page*)

Multiply by *m*th eigenfunction and weighting function [$p(x) = 1$ in this case] and integrate over interval of interest:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

As we have already seen, this leads to

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Apply IC #2: $$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty}\left[-A_n\frac{n\pi v}{L}(0) + B_n\frac{n\pi v}{L}(1)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$\text{or} \quad g(x) = \sum_{n=1}^{\infty} B_n\frac{n\pi v}{L}\sin\left(\frac{n\pi x}{L}\right)$$

Multiply by *m*th eigenfunction and weighting function [$p(x) = 1$ again] and integrate over interval of interest to obtain

$$B_n = \frac{2}{L}\cdot\frac{L}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \rightarrow \quad B_n = \frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

f. Full solution to the PDE is

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi vt}{L}\right)\right]$$

with $$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{and} \quad B_n = \frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

g. As with heat equation, the BCs typically determine the spatial eigenfunctions and the ICs typically determine the coefficients (using the orthogonality of the inner products). That is because the spatial problem is usually a BVP and the time problem is usually an IVP. EVPs are always BVPs.

h. Not modeled in this example:

i. Acoustic coupling between strings and nearby non-fixed objects
ii. Energy dissipation within string and nearby objects and due to air resistance
iii. Scattering (reflections) from nearby objects
iv. Presence of nearby objects, especially resonant cavities, that can affect sound perceived by listeners and can alter resonant frequencies somewhat
v. Gravity