

Lecture Outline for Monday, Oct. 5, 2026

1. Important theorems and concepts applicable to ODEs

- a. IVPs have unique solutions (not true for BVPs)
- b. Superposition principle; a linear combination of solutions to a homogeneous ODE over an interval is also a solution over the same interval
- c. Corollaries: 1) A constant multiple of a solution is also a solution; 2) homogeneous ODEs always possess at least the trivial solution $y = 0$
- d. Linearly independent vs. linearly dependent solutions (analogy to vectors)
- e. An N^{th} order homogeneous linear ODE has a fundamental set of N linearly independent solutions. The general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

- f. Nonhomogeneous ODEs: general solution = complementary solution + particular solution (complementary solution is the full solution set of the corresponding homogeneous ODE)
- g. Superposition also applies to particular solutions: If y_{p1} is a solution of the DE with $g_1(x)$, y_{p2} is a solution with $g_2(x)$, etc., then $y_{p1} + y_{p2} + \dots$ is a solution to

$$a_n(x) \frac{d^n y}{dx^n}(x) + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x) y = g_1(x) + g_2(x) + \cdots$$

2. Solution of nonhomogeneous linear ODEs with constant coefficients (not emphasized here)

- a. You can guess or...
- b. Method of undetermined coefficients (Sec. 3.4 of Zill, 6th ed.) – doesn't work for all forcing functions
- c. Method of variation of parameters (Sec. 3.5 of Zill, 6th ed.) – more general & complicated
- d. Use the annihilator method (see web link) – annihilators do not always exist

3. Solution strategies

- a. Identify class of ODE
- b. Applicable solution structure (What should solution look like?)
- c. Corresponding solution method
- d. Starting point: homogeneous solutions

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4. Review of finding independent solutions to ODEs

- a. Theorem 3.1.2 (Superposition Principle): If $y_1, y_2, \dots, y_k$ are solutions to a homogeneous n th-order differential equation over some interval along x , then the linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_k y_k(x)$, where c_1 through c_k are arbitrary constants, is also a solution over the same interval.
- b. Corollaries to Theorem 3.1.2:
 - i. A constant multiple $c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution (but linearly dependent).
 - ii. A homogeneous linear differential equation always possesses the trivial solution $y(x) = 0$.
- c. Two functions are *linearly independent* when neither is a constant multiple of the other over an interval. Implication: If solutions $y_1(x)$ and $y_2(x)$ to a homogeneous linear differential equation are linearly independent, then

$$\frac{y_2(x)}{y_1(x)} \neq \text{constant}; \text{ therefore, } \frac{y_2(x)}{y_1(x)} = u(x),$$

where $u(x)$ is some non-constant function of x over the interval; we'll apply this later.

- d. As with vectors, two solutions $y_1(x)$ and $y_2(x)$ might be *linearly independent* but not be *orthogonal*. Orthogonality will be defined later.
- e. Test for linear independence: Wronskian for two functions (can be extended to any number of functions; see Definition 3.1.2 in Zill, 6th ed.):

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

- f. Theorem 3.1.3 (Linearly independent solutions): If $y_1, y_2, \dots, y_n$ are solutions to a homogeneous n th-order differential equation over some interval along x , then the set of solutions is linearly independent over that interval iff $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval.

5. Important special case: Solution of homogeneous 2nd-order linear differential equations with constant coefficients

- a. General form

$$ay'' + by' + cy = 0$$

- b. A likely solution is e^{rx} , where r is a constant, because any number of derivatives of e^{rx} is a constant times e^{rx} . Substitute the likely solution into the diff eq:

$$a \frac{d^2(e^{rx})}{dx^2} + b \frac{d(e^{rx})}{dx} + ce^{rx} = 0 \rightarrow ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0 \rightarrow (ar^2 + br + c)e^{rx} = 0$$

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- c. Since e^{rx} cannot be equal to zero for any value of x , it must be true that

$$ar^2 + br + c = 0 \rightarrow r = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$$

- d. This is called the *auxiliary equation* or *characteristic equation* of the diff eq. There are three possible types of roots:

- i. Distinct real roots: $r = r_1$ and r_2 , and the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

- ii. Complex conjugate roots: $r = \alpha \pm i\beta$, and the general solution is

$$y(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

Using Euler's formula,

$$e^{i\theta} = \cos \theta + i \sin \theta ,$$

an alternative (and often more useful) form of the general solution is

$$y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

- iii. Repeated real roots: $r = r_1$ and r_1 (same values); a special approach (the reduction of order method) is needed to find linearly independent solutions. We already know that $\exp(r_1 x)$ is a solution, but we need a second independent solution. Recall that the second solution must satisfy

$$\frac{y_2(x)}{y_1(x)} = u(x) \rightarrow y_2(x) = u(x) y_1(x)$$

Substitute $y_2 = uy_1$ into the original diff eq:

$$\begin{aligned} a(uy_1)'' + b(uy_1)' + cuy_1 &= 0 \rightarrow a(uy_1' + u'y_1)' + b(uy_1' + u'y_1) + cuy_1 = 0 \\ \rightarrow a(uy_1'' + 2u'y_1' + u''y_1) + b(uy_1' + u'y_1) + cuy_1 &= 0 \\ \rightarrow auy_1'' + buy_1' + cuy_1 + a(2u'y_1' + u''y_1) + b(u'y_1) &= 0 \\ \rightarrow u(ay_1'' + by_1' + cy_1) + a(2u'y_1' + u''y_1) + b(u'y_1) &= 0 \\ \rightarrow u(0) + 2au'y_1' + au''y_1 + bu'y_1 &= 0 \end{aligned}$$

because $ay_1'' + by_1' + cy_1 = 0$ if y_1 is a solution.

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Since there are only derivatives of the function u in the last line above, we can reduce the order of the derivatives (hence the name of this method) by defining a second new function $w(x) = u'(x)$, substituting it into the result immediately above, and then dividing by wy_1 :

$$2awy_1' + aw'y_1 + bwy_1 = 0 \rightarrow 2a \frac{y_1'}{y_1} + a \frac{w'}{w} + b = 0$$

Substituting $w' = dw/dx$ and $y_1' = dy_1/dx$, rearranging, and integrating yields

$$\begin{aligned} a \frac{1}{w} \cdot \frac{dw}{dx} + 2a \frac{1}{y_1} \cdot \frac{dy_1}{dx} + b &= 0 \rightarrow a \cdot \frac{dw}{w} + 2a \frac{dy_1}{y_1} + b dx = 0 \\ \rightarrow a \int \frac{dw}{w} + 2a \int \frac{dy_1}{y_1} dx + b \int dx + c &= 0 \\ \rightarrow a \ln w + 2a \ln y_1 + bx + c &= 0 \\ \rightarrow a \ln w + a \ln y_1^2 + bx + c &= 0 \\ \rightarrow \ln(wy_1^2) = -\frac{b}{a}x + c_1 & \\ \rightarrow wy_1^2 = e^{-\frac{b}{a}x + c_1} = e^{-\frac{b}{a}x} e^{c_1} = c_2 e^{-\frac{b}{a}x} & \end{aligned}$$

Substitute $w = u'$ back in and integrate again to obtain

$$u'y_1^2 = c_2 e^{-\frac{b}{a}x} \rightarrow u' = c_2 \frac{e^{-\frac{b}{a}x}}{y_1^2} \rightarrow u = c_2 \int \frac{e^{-\frac{b}{a}x}}{y_1^2} dx + c_3$$

We are free to choose $c_2 = 1$ and $c_2 = 0$. (We will still get a linearly independent second solution with those choices.) Recalling that $y_2 = uy_1$ yields the result

$$y_2(x) = y_1(x) \int \frac{e^{-\frac{b}{a}x}}{y_1^2(x)} dx$$

See Sec. 3.2 to see how the reduction of order method is applied for the more general case when $y'' + P(x)y' + Q(x)y = 0$.

Getting back to the original case of repeated real roots, $r = r_1, r_1$, there appears to be only the one solution $y_1(x) = \exp(r_1x)$. However, we can now obtain a second linearly independent solution using the newly derived formula with the substitution $y_1(x) = \exp(r_1x)$:

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$$y_2(x) = e^{r_1 x} \int \frac{e^{-\frac{b}{a}x}}{(e^{r_1 x})^2} dx = e^{r_1 x} \int \frac{e^{-\frac{b}{a}x}}{e^{2r_1 x}} dx$$

Recall that the roots are related to the coefficients in the original diff eq via the quadratic formula. Because there are repeated real roots, we know that $b^2 - 4ac = 0$, so

$$r_1 = -\frac{b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac} = -\frac{b}{2a} \pm \frac{1}{2a}(0) \rightarrow -\frac{b}{a} = 2r_1$$

Substituting this result into the expression for $y_2(x)$ yields

$$y_2(x) = e^{r_1 x} \int \frac{e^{2r_1 x}}{e^{2r_1 x}} dx = e^{r_1 x} \int dx = x e^{r_1 x}$$

Thus, for a 2nd-order linear differential equation with constant coefficients that has a characteristic equation with repeated roots, the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

6. Solutions to Fourier and modified Fourier equations:

$$y'' + a^2 y = 0 \quad \text{and} \quad y'' - a^2 y = 0$$

- a. For closed boundaries (i.e., problem defined over finite range of x), the recommended solutions are

$$\text{Fourier equation: } y(x) = c_1 \cos(ax) + c_2 \sin(ax)$$

$$\text{Modified Fourier equation: } y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

Roots r_1 and r_2 of characteristic equation are imaginary for the Fourier equation and real for the modified Fourier equation. In both cases, the roots are distinct.

- b. For open boundaries (i.e., problem defined over infinite or semi-infinite range of x), the recommended solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$