ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Friday, Nov. 6, 2026

1. Interpretation of wave equation solution

    a. What can we learn from it?
    b. Vibration modes and resonances; harmonically related
    c. *Matlab* simulation
    d. Standing waves vs. traveling waves. For example, consider the $g(x) = 0$ (or $B_n = 0$) case. Each eigenmode, which is a *standing* wave, can be expressed as the sum of two counterpropagating *traveling* waves with speed v. Use the identity

$$\sin(a)\cos(b) = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$$

    to obtain

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) = \frac{1}{2}\sin\left[\frac{n\pi}{L}(x+vt)\right] + \frac{1}{2}\sin\left[\frac{n\pi}{L}(x-vt)\right]$$

    All eigenmodes propagate at the same speed v in this simple example. In more realistic problems, different eigenmodes travel at different speeds (dispersion).

2. Wave equation (1-D) problems with *open* boundaries (or boundaries so far away that they can be considered to be infinite in extent)

$$v^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad -\infty \le x \le \infty, \quad t \ge 0$$

$$\text{ICs:} \quad u(x,0) = f(x) \quad \text{and} \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$

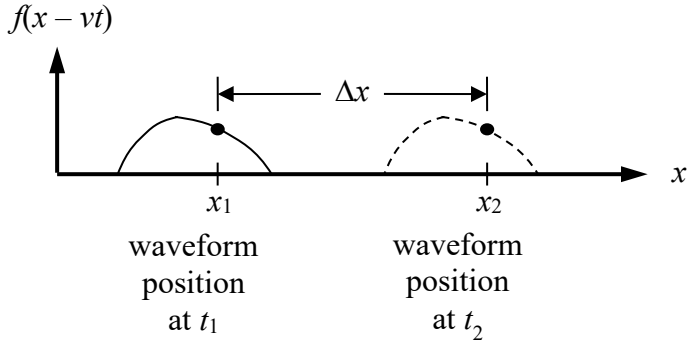
    a. Special solution methods required (e.g., Green's functions for nonhomogeneous problems)
    b. One approach: See supplemental reading on D'Alembert's solution
    c. Solution has $x \pm vt$ in arguments of functions (traveling waves)

$$u(x,t) = \frac{1}{2}f(x+vt) + \frac{1}{2}f(x-vt) + \frac{1}{2v}g_A(x+vt) - \frac{1}{2v}g_A(x-vt),$$

    where $g_A(x)$ is the antiderivative (integral) of $g(x)$.

(*continued on next page*)

d. Example: depiction of propagation of the waveform $f(x - vt)$ over time:



The point on the waveform marked with a dot is at $x = x_1$ at time t_1. At later time t_2, the waveform advances to the right so that the dot reaches the location $x = x_2$. The waveform travels the distance $\Delta x = x_2 - x_1$ over the time interval $\Delta t = t_2 - t_1$. The dot represents a particular point along the function $f(x)$, so

$$x_2 - vt_2 = x_1 - vt_1 \quad \rightarrow \quad x_2 - x_1 = v(t_2 - t_1) \quad \rightarrow \quad \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = v \qquad (v = \text{speed})$$

3. Wave equation in polar (cylindrical) coordinate system. Example: Vibrating circular membrane of radius c, assuming only radial vibrational modes (no angle-dependent modes):

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad 0 \le r \le c \quad \text{and} \quad t \ge 0$$

a. ODEs after separation (R equation is a parametric Bessel equation of order 0)

$$r^2 R'' + rR' + \lambda r^2 R = 0 \quad \text{and} \quad T'' + \lambda a^2 T = 0$$

b. BCs and ICs and their interpretation (singularity at $r = 0$, so no BC there; see self-adjoint form of parametric Bessel equation in Oct. 23 lecture notes):

$$u(c,t) = 0, \quad u(r,0) = f(r), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r); \quad u \text{ is finite everywhere}$$

a. Suggested solution forms for ODEs (with $\lambda_n = \alpha_n^2$)

$$R_n(r) = c_1 J_0(\alpha_n r) + c_2 Y_0(\alpha_n r) \quad \text{and} \quad T_n(t) = c_3 \cos(a\alpha_n t) + c_4 \sin(a\alpha_n t)$$

b. The $R(r)$ problem is an eigenvalue problem (it's also a BVP) and specifically a singular Sturm-Liouville problem; the $T(t)$ problem is an IVP
c. Special additional condition: Solution must be finite within boundary (for $r \le c$, which includes $r = 0$), so $Y_0(\alpha_n r)$ solution must be eliminated.
d. In the case of a drum being struck by a stick or mallet, $f(r) = 0$ and $g(r)$ is a pulse centered at $r = 0$.