

Lecture Outline for Wednesday, Oct. 7, 2026

1. Solution of homogeneous 2nd-order linear differential equations with constant coefficients (continued): repeated roots in auxiliary equation

Use the reduction of order method to find two linearly independent solutions. One solution is $y_1(x) = \exp(r_1 x)$. The second solution must satisfy

$$y_2(x) = u(x) y_1(x)$$

Substitute $y_2 = uy_1$ into the original diff eq:

$$\begin{aligned} a(uy_1)'' + b(uy_1)' + cuy_1 &= 0 \\ u(ay_1'' + by_1' + cy_1) + a(2u'y_1 + u''y_1) + b(u'y_1) &= 0 \\ u(0) + 2au'y_1 + au''y_1 + bu'y_1 &= 0 \end{aligned}$$

because $ay_1'' + by_1' + cy_1 = 0$ if y_1 is a solution.

There are only derivatives of the function u in the last line above, so we can reduce the order of the derivatives (hence the name “reduction of order” method) by defining a second new function $w(x) = u'(x)$. Substitute and divide by wy_1 :

$$2awy_1' + aw'y_1 + bwy_1 = 0 \quad \rightarrow \quad 2a \frac{y_1'}{y_1} + a \frac{w'}{w} + b = 0$$

Substituting $w' = dw/dx$ and $y_1' = dy_1/dx$, rearranging, and integrating yields

$$\begin{aligned} a \frac{1}{w} \cdot \frac{dw}{dx} + 2a \frac{1}{y_1} \cdot \frac{dy_1}{dx} + b &= 0 \quad \rightarrow \quad a \cdot \frac{dw}{w} + 2a \frac{dy_1}{y_1} + b dx = 0 \\ \rightarrow a \int \frac{dw}{w} + 2a \int \frac{dy_1}{y_1} + b \int dx + c &= 0 \\ \rightarrow a \ln w + 2a \ln y_1 + bx + c &= 0 \\ \rightarrow a \ln w + a \ln y_1^2 + bx + c &= 0 \\ \rightarrow \ln(wy_1^2) = -\frac{b}{a}x + c_1 & \\ \rightarrow wy_1^2 = e^{-\frac{b}{a}x + c_1} = e^{-\frac{b}{a}x} e^{c_1} = c_2 e^{-\frac{b}{a}x} & \end{aligned}$$

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Substitute $w = u'$ back in and integrate again to obtain

$$u'y_1^2 = c_2 e^{-\frac{b}{a}x} \rightarrow u' = c_2 \frac{e^{-\frac{b}{a}x}}{y_1^2} \rightarrow u = c_2 \int \frac{e^{-\frac{b}{a}x}}{y_1^2} dx + c_3$$

We are free to choose $c_2 = 1$ and $c_3 = 0$. (We will still get a linearly independent second solution with those choices.) Recalling that $y_2 = uy_1$ yields the result

$$y_2(x) = y_1(x) \int \frac{e^{-\frac{b}{a}x}}{y_1^2(x)} dx$$

See Sec. 3.2 to see how the reduction of order method is applied for the more general case when $y'' + P(x)y' + Q(x)y = 0$.

Getting back to the original case of repeated real roots, $r = r_1, r_1$, we can now obtain a second linearly independent solution using the newly derived formula with the substitution $y_1(x) = \exp(r_1x)$:

$$y_2(x) = e^{r_1x} \int \frac{e^{-\frac{b}{a}x}}{(e^{r_1x})^2} dx = e^{r_1x} \int \frac{e^{-\frac{b}{a}x}}{e^{2r_1x}} dx$$

Recall that the roots are related to the coefficients in the original diff eq via the quadratic formula. Because there are repeated real roots, we know that $b^2 - 4ac = 0$, so

$$r_1 = -\frac{b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac} = -\frac{b}{2a} \pm \frac{1}{2a}(0) \rightarrow -\frac{b}{a} = 2r_1$$

Substituting this result into the expression for $y_2(x)$ yields

$$y_2(x) = e^{r_1x} \int \frac{e^{2r_1x}}{e^{2r_1x}} dx = e^{r_1x} \int dx = xe^{r_1x}$$

Thus, for a homogeneous 2nd-order linear differential equation with constant coefficients that has a characteristic equation with repeated roots, the general solution is

$$y(x) = c_1 e^{r_1x} + c_2 x e^{r_1x}$$

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2. Solutions to Fourier and modified Fourier equations:

$$y'' + a^2 y = 0 \quad \text{and} \quad y'' - a^2 y = 0$$

- a. For closed boundaries (i.e., problem defined over finite range of x), the recommended solutions are

$$\text{Fourier equation: } y(x) = c_1 \cos(ax) + c_2 \sin(ax)$$

$$\text{Modified Fourier equation: } y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

Roots r_1 and r_2 of characteristic equation are imaginary for the Fourier equation and real for the modified Fourier equation. In both cases, the roots are distinct.

- b. For open boundaries (i.e., problem defined over infinite or semi-infinite range of x), the recommended solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

whether the roots r_1 and r_2 are real or imaginary.

3. Example #1: BVP involving Fourier equation:

$$y'' + a^2 y = 0 \quad \text{with} \quad y(0) = 0, \quad y(1) = 0$$

- Roots of auxiliary equation: $r = \pm ia$ ($\alpha = 0$ and $\beta = a$)
- Closed boundary, so general solution is $y(x) = c_1 \cos(ax) + c_2 \sin(ax)$
- Apply boundary conditions:

$$y(0) = 0 = c_1 \cos(a \cdot 0) + c_2 \sin(a \cdot 0) \rightarrow c_1 = 0$$

$$y(1) = 0 = c_2 \sin(a) \rightarrow a_n = n\pi \quad \text{for } n = 1, 2, 3, \dots$$

- Nontrivial solution is $y(x) = c_2 \sin(n\pi x)$, $n = 1, 2, 3, \dots$
- Infinitely many nontrivial solutions since infinitely many integers n will work. This is an eigenvalue problem. Constants $a_n = n\pi$ are *eigenvalues*, and elementary solutions $\sin(n\pi x)$ are *eigenfunctions*.
- Compare $y'' = -a^2 y$ to $Ay = \lambda y$, where A is a linear operator. (A 2nd order derivative is also a linear operator.)
- The constant c_2 is left unspecified in this problem. However, if there had been a forcing function [i.e., $y'' + a^2 y = g(x)$], then c_2 could be uniquely specified.