

Lecture Outline for Friday, Oct. 9, 2026

1. Example #1: BVP involving Fourier equation:

$$y'' + a^2 y = 0 \quad \text{with} \quad y(0) = 0, \quad y(1) = 0$$

- a. Roots of auxiliary equation: $r = \pm ia$ ($\alpha = 0$ and $\beta = a$)
- b. Closed boundary, so recommended general solution is $y(x) = c_1 \cos(ax) + c_2 \sin(ax)$
- c. Apply boundary conditions:

$$y(0) = 0 = c_1 \cos(a \cdot 0) + c_2 \sin(a \cdot 0) \rightarrow c_1 = 0$$

$$y(1) = 0 = c_2 \sin(a) \rightarrow a_n = n\pi \quad \text{for } n = 1, 2, 3, \dots$$

- d. Nontrivial solution is $y(x) = c_2 \sin(n\pi x)$, $n = 1, 2, 3, \dots$
- e. Infinitely many nontrivial solutions since infinitely many integers n will work. This is an eigenvalue problem. Constants $a_n = n\pi$ are *eigenvalues*, and elementary solutions $\sin(n\pi x)$ are *eigenfunctions*.
- f. Compare $y'' = -a^2 y$ to $Ay = \lambda y$, where A is a linear operator. (A 2nd order derivative is also a linear operator.)
- g. The constant c_2 is left unspecified in this problem. However, if there had been a forcing function [i.e., $y'' + a^2 y = g(x)$], then c_2 could be uniquely specified.

2. Example #2: Now consider a specific value for a^2 but the same BCs:

$$y'' + 1.69\pi^2 y = 0 \quad \text{with} \quad y(0) = 0, \quad y(1) = 0$$

- a. Recommended general solution is $y(x) = c_1 \cos(1.3\pi x) + c_2 \sin(1.3\pi x)$
- b. Apply boundary conditions:

$$y(0) = 0 = c_1 \cos(1.3\pi \cdot 0) + c_2 \sin(1.3\pi \cdot 0) \rightarrow c_1 = 0$$

$$y(1) = 0 = c_2 \sin(1.3\pi) = c_2 (-0.809) \rightarrow c_2 = 0$$

- c. No nontrivial solutions; only the trivial solution $y = 0$.
- d. Not all BVPs have nontrivial solutions.

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3. Example #3: BVP involving modified Fourier equation:

$$y'' - a^2 y = 0 \quad \text{with} \quad y(0) = 0, \quad y(1) = 0$$

- a. Recommended general solution is $y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$
- b. Apply boundary conditions:

$$y(0) = 0 = c_1 \cosh(a \cdot 0) + c_2 \sinh(a \cdot 0) \rightarrow c_1 = 0$$

$$y(1) = 0 = c_2 \sinh(a) \rightarrow c_2 = 0$$

The $\sinh(x)$ function is not periodic; it equals zero only at $x = 0$.
See Wikipedia article “Hyperbolic functions” for plots of $\sinh(x)$ and $\cosh(x)$.

- c. No nontrivial solutions; only the trivial solution $y = 0$.

4. Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0$$

$$\text{special case: } x^2 y'' + xy' - \alpha^2 y = 0 \quad (a = b = 1 \text{ and } c \text{ neg.})$$

$$\text{special special case: } x^2 y'' + xy' - y = 0 \quad (a = b = 1 \text{ and } c = -1)$$

5. “Peel-the-onion” method applied to Cauchy-Euler equation with $a = b = 1$ and $c = -1$

$$x^2 y'' + xy' - y = 0 \quad \text{equiv. to} \quad \frac{d}{dx} \left[\frac{1}{x} \frac{d}{dx} (xy) \right] = 0$$

Successive integrations to arrive at solution. First integration w.r.t. x :

$$\frac{1}{x} \frac{d}{dx} (xy) = c_1 \rightarrow \frac{d}{dx} (xy) = c_1 x$$

Second integration w.r.t. x :

$$xy = c_1 \frac{x^2}{2} + c_2 \rightarrow y = c_1 \frac{x}{2} + c_2 \frac{1}{x}$$

6. Solution to 2nd order Cauchy-Euler equation with $a = b = 1$ and c neg. (See Sec. 3.6 of Zill, 6th ed.; see also Sec. 3.2, which explains the reduction of order method for finding a second independent solution)

$$x^2 y'' + xy' - \alpha^2 y = 0 \quad \text{solutions are} \quad y = \begin{cases} c_1 + c_2 \ln x, & \alpha = 0 \\ c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha > 0 \end{cases}$$