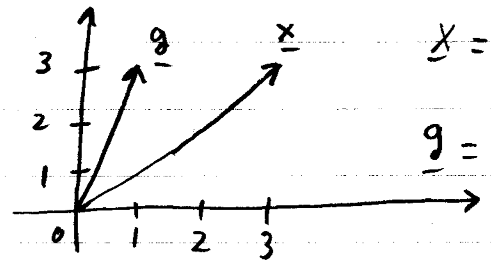
Vectors, Signals, and Fourier series

Vectors:



$\underline{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$\underline{g} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{g} = c\,\underline{x} + \underline{e}$: The "error" vector $\underline{e}$ is a function of c.

What value of c gives the smallest <u>length</u> of $\underline{e}$?

"Length" of a vector : $|\underline{x}| = \sqrt{3^2 + 3^2} = 3 \cdot \sqrt{2}$

$|\underline{g}| = \sqrt{1^2 + 3^2} = \sqrt{10}$

"Dot product" or "scalar product" or "inner product":

$(\underline{x}, \underline{g}) = \underline{x} \cdot \underline{g} = 3 \cdot 1 + 3 \cdot 3 = 12$

Some properties: Dot product is <u>linear</u>:

For scalar a, $(a\underline{x}, \underline{g}) = (\underline{x}, a\underline{g}) = a(\underline{x}, \underline{g})$

$(\underline{x}, \underline{g}_1 + \underline{g}_2) = (\underline{x}, \underline{g}_1) + (\underline{x}, \underline{g}_2)$

Length of $\underline{e}$ as a function of c:

$|\underline{e}(c)|^2 = \underline{e}(c) \cdot \underline{e}(c) = [\underline{g} - c\underline{x}] \cdot [\underline{g} - c\underline{x}]$

$= |\underline{g}|^2 + c^2 |\underline{x}|^2 - 2c\ \underline{x} \cdot \underline{g}$

What value of c <u>minimizes</u> $|\underline{e}(c)|^2$?

$\frac{d}{dc} |\underline{e}(c)|^2 = 0 \quad \Rightarrow \quad c = \frac{\underline{x} \cdot \underline{g}}{|\underline{x}|^2} = \frac{12}{18} = \frac{2}{3}$

<u>Def</u>: vectors are <u>orthogonal</u> if $\underline{x} \cdot \underline{g} = 0$.

Equivalent condition for minimum $|\underline{e}(c)|^2$:

c $\underline{x}$ is the orthogonal projection of $\underline{g}$ on $\underline{x}$.

$$\left\{ \begin{array}{l} \underline{e} \cdot \underline{x} = 0 \\ [\underline{g} - c\,\underline{x}] \cdot \underline{x} = 0 \\ \underline{g} \cdot \underline{x} - c|\underline{x}|^2 = 0 \quad \Rightarrow \quad c = \dfrac{\underline{g} \cdot \underline{x}}{|\underline{x}|^2} \end{array} \right.$$

<u>Signals</u>:

"Length" of $g(t)$ is $\sqrt{E_g}$, $E_g = \int_{-\infty}^{\infty} g(t)^2\,dt$

Inner product of two signals $x(t)$, $g(t)$:

$$(x, g) = \int_{-\infty}^{\infty} x(t)\,g(t)\,dt$$

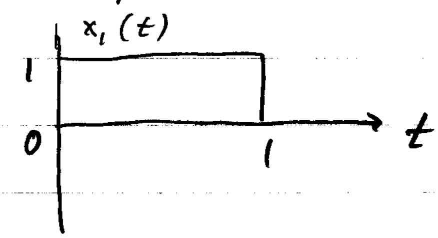
Note $E_g = (g, g)$

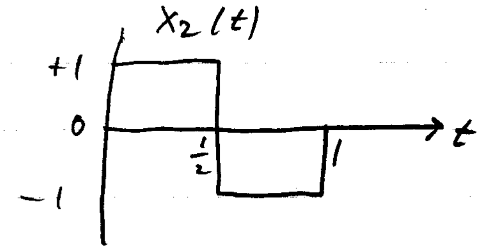
How to choose scalar c to approximate a signal $g(t)$ with another signal $x(t)$ so that the "error energy" is minimum?

$$g(t) = c\,x(t) + e(t)$$

<u>Answer</u>: $c =$

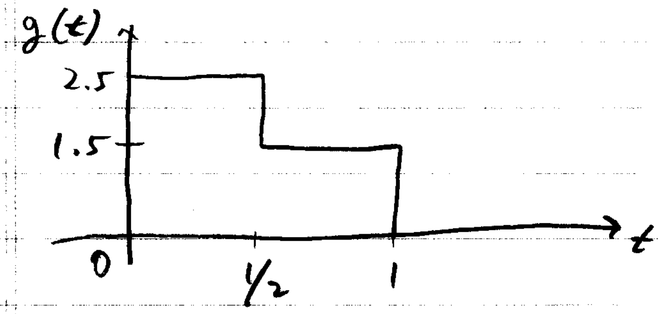
Example: Simple "wavelet"





Are $x_1(t)$ and $x_2(t)$ orthogonal? ______

$$(x_1, x_2) = \int_0^1 x_1(t)\, x_2(t)\, dt =$$



"Best" approximation of $g(t)$ with $x_1(t)$:

$$g(t) = c_1 x_1(t) + e_1(t)$$

$$\Rightarrow c_1 = \frac{\int_0^1 g(t)\, x_1(t)\, dt}{\int_0^1 x_1(t)^2\, dt} = \text{\_\_\_\_\_\_}$$

"Best" approximation of $g(t)$ with $x_2(t)$:

$$g(t) = c_2 x_2(t) + e_2(t)$$

$$\Rightarrow c_2 = \frac{\int_0^1 g(t)\, x_2(t)\, dt}{\int_0^1 x_2(t)^2\, dt} = \text{\_\_\_\_\_\_}$$

Note for this case,

$g(t) = \boxed{}\; x_1(t) + \boxed{}\; x_2(t)$ is exact (no error).

( )

Exponential Fourier series:

Energy and inner product of complex signals:

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = (g, g)$$

$$(g, x) = \int_{-\infty}^{\infty} g(t)\, x(t)^* dt$$

Fourier series for periodic $g(t)$ with period T_0:

We use "basis functions" that are complex exponentials:

$$x_n(t) = e^{jn\omega_0 t}, \quad n = 0, \pm 1, \pm 2, \dots, \qquad \omega_0 = \frac{2\pi}{T_0}$$

These are orthogonal over one period T_0:

$$(x_n, x_m) = \int_{T_0} x_n(t)\, x_m(t)^* dt$$

$$= \int_{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = \begin{cases} 0, & n \neq m \\ T_0, & n = m \end{cases}$$

The set $\{ e^{jn\omega_0 t}, \; n = 0, \pm 1, \pm 2, \dots \}$ is also "complete".

How do we choose D_n, $n = 0, \pm 1, \pm 2, \dots$ so that

$$g(t) = \sum_{n=-\infty}^{\infty} D_n\, x_n(t) = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\omega_0 t} \; ?$$

$$\Rightarrow \quad D_n = \frac{(g, x_n)}{(x_n, x_n)} = \frac{1}{T_0} \int_{T_0} g(t)\, e^{-jn\omega_0 t} dt$$

Other forms for Fourier series:

$$C_n \cos[n\omega_0 t + \theta_n] = \underbrace{\left(\frac{C_n}{2} e^{j\theta_n}\right)}_{D_n} e^{jn\omega_0 t} + \underbrace{\left(\frac{C_n}{2} e^{-j\theta_n}\right)}_{D_{-n}} e^{-jn\omega_0 t}$$

$$= \underbrace{[C_n \cos\theta_n]}_{a_n} \cos(n\omega_0 t) + \underbrace{[-C_n \sin\theta_n]}_{b_n} \sin(n\omega_0 t)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

Fourier spectra for $g(t) = 7 + 40 \cos(2\pi 1000 t + \pi/3)$

"One-sided" spectrum:

$\omega_0 =$

$C_0 =$

$C_1 =$

$\theta_1 =$

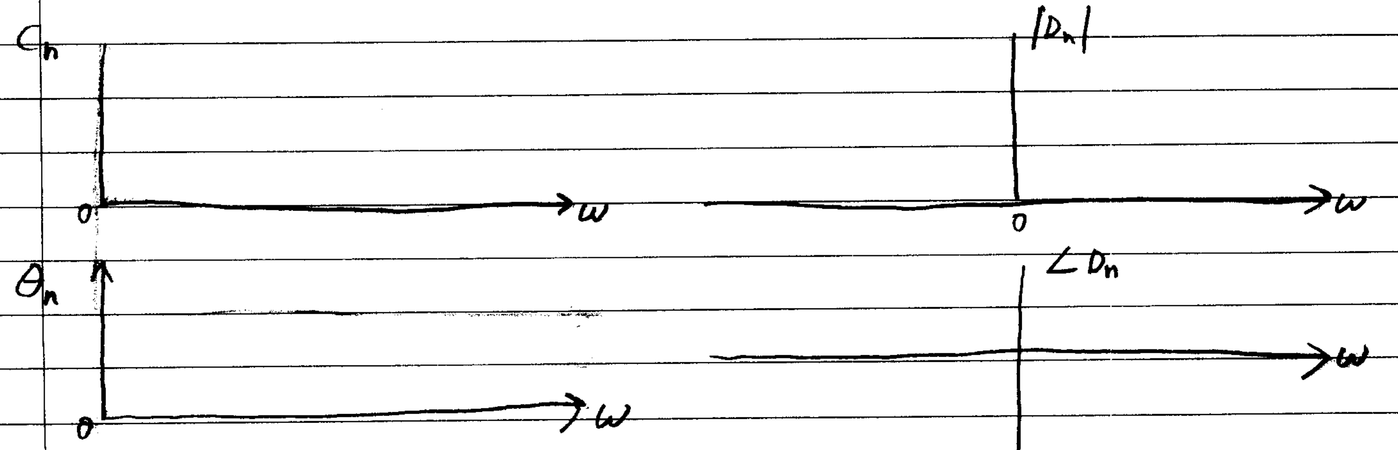
"Two-sided" spectrum:

$\omega_0 =$

$D_0 =$

$D_1 =$

$D_{-1} =$



What is "negative frequency"?

$$\cos(2\pi\, 1000 t) = \frac{1}{2} e^{j 2\pi\, 1000 t} + \frac{1}{2} e^{-j 2\pi\, 1000 t}$$

Real sinusoid has "+" and "−" "frequencies".
$\Rightarrow$ Amplitude is split in half between components.

In modulation, negative frequencies become real & observable!
(we'll see later)

Fourier series computation:

See Examples