ELEC 320
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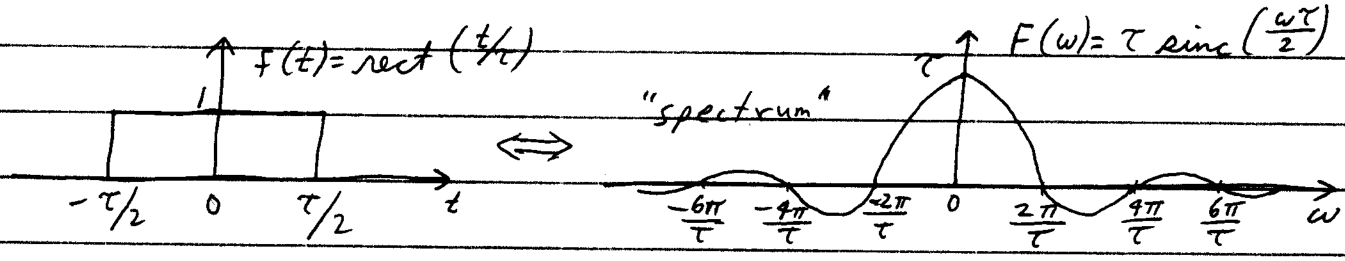
FOURIER TRANSFORM

The Fourier transform represents non-periodic signals in the frequency domain, i.e., as a sum of sinusoids with properly-chosen frequencies, amplitudes, and phases.

Ex:



- Note the "spectrum" $F(\omega)$ contains a continuum of frequencies, in contrast to the discrete frequencies $\omega_0, 2\omega_0, 3\omega_0, \ldots$ in the Fourier series.

- The spectrum is 2-sided, with "negative frequencies" interpreted as we discussed with the exponential Fourier series:

$$\cos(\omega t) = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$$

- **Fourier transform definition:**

Inverse Fourier tr. $\rightarrow$ $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

(Forward) Fourier tr. $\rightarrow$ $$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt$$

Exponential Fourier series:

$$f_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{j(n\omega_0)t}$$

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-j(n\omega_0)t} \, dt$$

where $\omega_0 = \frac{2\pi}{T_0}$.

- Get transform from series by $T_0 \rightarrow \infty$. (see text).

- Notation: $f(t) \Leftrightarrow F(\omega)$; $F(\omega) = \mathcal{F}\{f(t)\}$

$f(t) = \mathcal{F}^{-1}\{F(\omega)\}$

$F(\omega)$ is generally complex-valued, even when $f(t)$ is real:

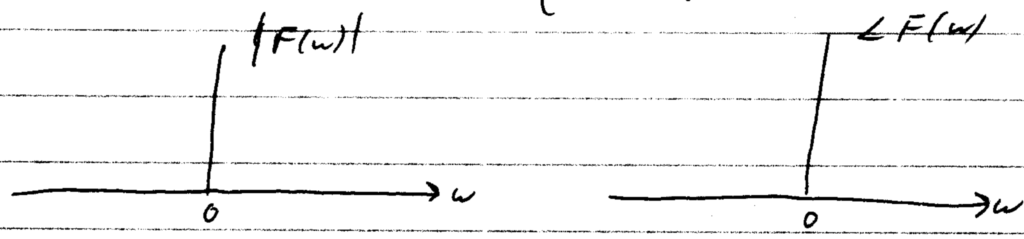
$$F(\omega) = |F(\omega)|\, e^{j\angle F(\omega)}$$

Amplitude or magnitude spectrum

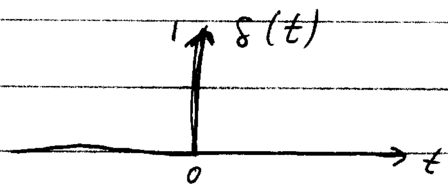
Phase spectrum

Symmetry properties when $f(t)$ is <u>real</u>-valued:

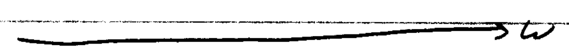
$$F(-\omega) = F^*(\omega) \Rightarrow \begin{cases} |F(-\omega)| = \\ \angle F(-\omega) = \end{cases}$$

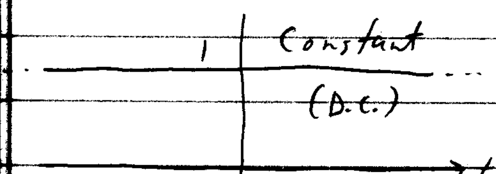


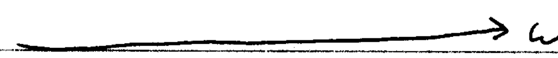
Some Fourier transform pairs:

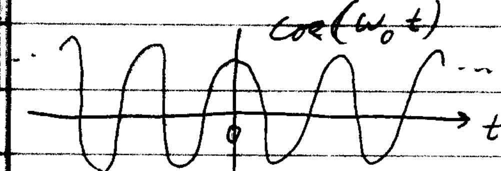


$\Leftrightarrow$





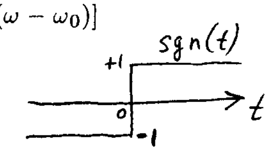




$3\cos\left(100t + \frac{\pi}{3}\right) \Leftrightarrow$

A Short Table of Fourier Transforms

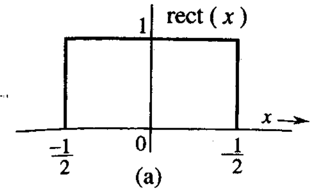
	$f(t)$	$F(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

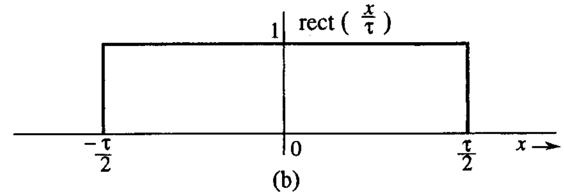


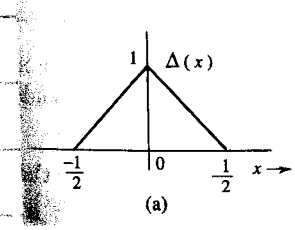
$\omega_0 = \dfrac{2\pi}{T}$

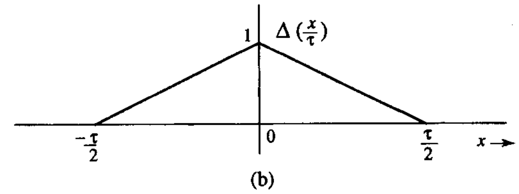
Note:
A more common definition is

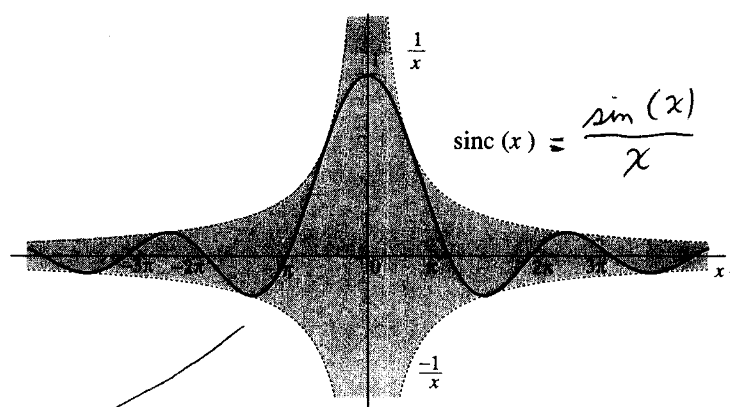
$\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$











sinc (x) $= \dfrac{\sin(x)}{x}$

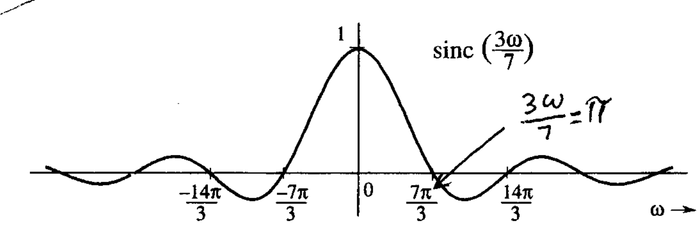
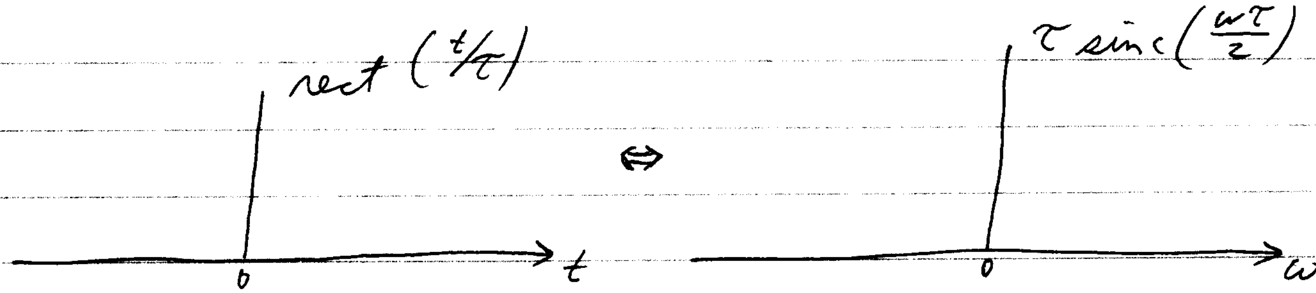
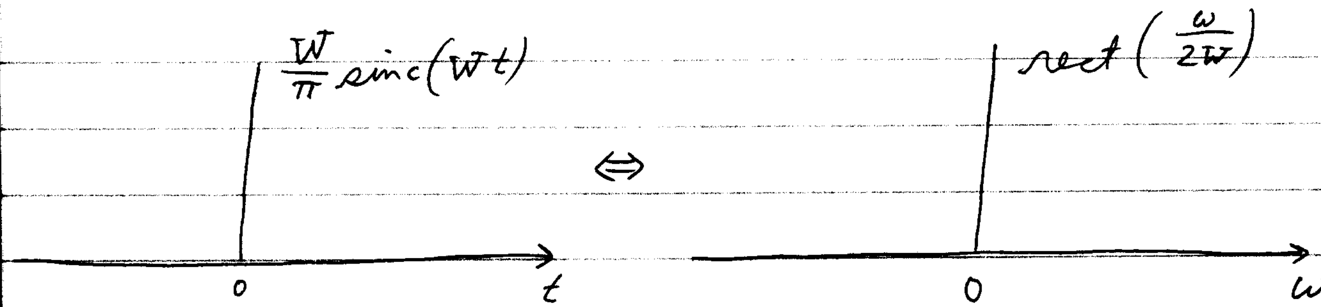


Fig. 4.9 A sinc pulse.

Some important F.T. pairs:





<u>Notes:</u>

- Signal thinner in one domain $\Leftrightarrow$ wider in other domain.
- What is the "time duration" and "bandwidth" of each signal above?

<u>Exercises:</u>

- Show $\mathcal{F}\{rect(t)\} = sinc\left(\frac{\omega}{2}\right)$
- Sketch the following, and find the forward (or inverse) Fourier transform.

$7 \, rect\left(\frac{t}{.001}\right)$ $\qquad$ $sinc(2\omega)$

$3 \, sinc(1000t)$ $\qquad$ $sinc\left(\frac{\omega}{2}\right)$