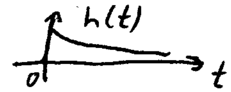
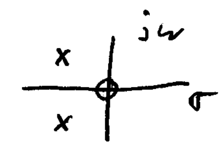
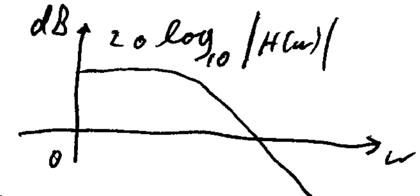
Relationships between various ways to characterize the zero-state response (ZSR) of linear, time-invariant, continuous-time (LTIC) systems

① Impulse response $h(t)$ 

② Transfer function $H(s) = \mathcal{L}\{h(t)\}$: Poles & zeros 

③ Frequency response $H(\omega) = \mathcal{F}\{h(t)\}$: Bode plot 

$$= H(s)\Big|_{s=j\omega}$$

Time domain: $\left[\ \delta(t),\ x(t)\ \right] \longrightarrow$ **LTIC system** $\begin{pmatrix} h(t) \\ H(\omega) \\ H(s) \end{pmatrix}$ $\longrightarrow \left[\ h(t),\ y(t) = x(t) * h(t)\ \right]$

Frequency domain: $\left[\ A\cos\omega_0 t,\ X(\omega) = \mathcal{F}\{x(t)\}\ \right]$ → $\left[\ |H(\omega_0)| \cdot A\cos[\omega_0 t + \angle H(\omega_0)],\ Y(\omega) = X(\omega)\cdot H(\omega)\ \right]$ $\boxed{\text{Signal Process., Communication}}$

s-domain: $\left[\ X(s) = \mathcal{L}\{x(t)\}\ \right]$ → $\left[\ Y(s) = X(s)\cdot H(s)\ \right]$ $\boxed{\text{Control Systems}}$

Frequency Response & Transfer Function

Recall that the frequency response of a system tells the amplitude gain and phase shift experienced by a cosine:

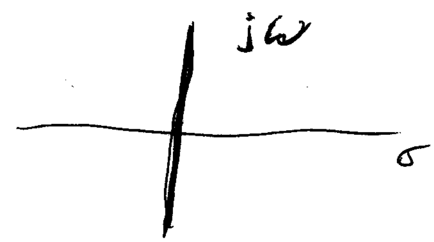
$$2\cos(1000t) \xrightarrow{\quad} \boxed{\underset{\text{System}}{H(\omega)}} \xrightarrow{\quad} 2\cdot|H(1000)|\cos\left(1000t + \angle H(1000)\right)$$

Recall transfer function:

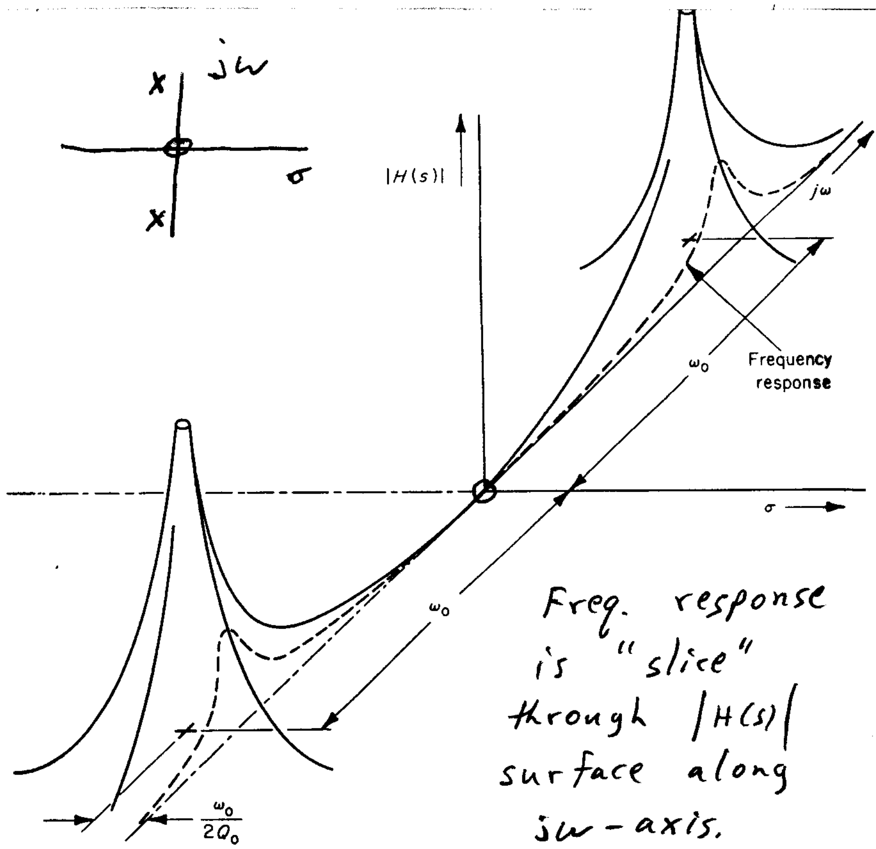
$$H(s) = \frac{\mathcal{L}\{\text{output signal}\}}{\mathcal{L}\{\text{input signal}\}} = \mathcal{L}\{\text{impulse response}\}$$

The frequency response is obtained by evaluating the transfer function along the imaginary axis $s = j\omega$ in the s-plane:

$$H(\omega) = H(s)\Big|_{s=j\omega}$$



(*) This provides insight for the design of analog filters.



Freq. response is "slice" through $|H(s)|$ surface along $j\omega$-axis.

Fig. 5.26 *A three-dimensional view of $|H(s)|$ versus s, with the dashed line showing the intersection of the plane $\sigma = 0$ and the surface $|H(s)|$, thus placing the frequency response in evidence.*

What is the effect on the frequency response of a:

- Zero on the $j\omega$-axis?

- Pole(s) near the $j\omega$ axis?

- Where should pole be placed for a <u>narrow</u> band-pass filter?

Geometric evaluation of Frequency response from pole-zero plot:

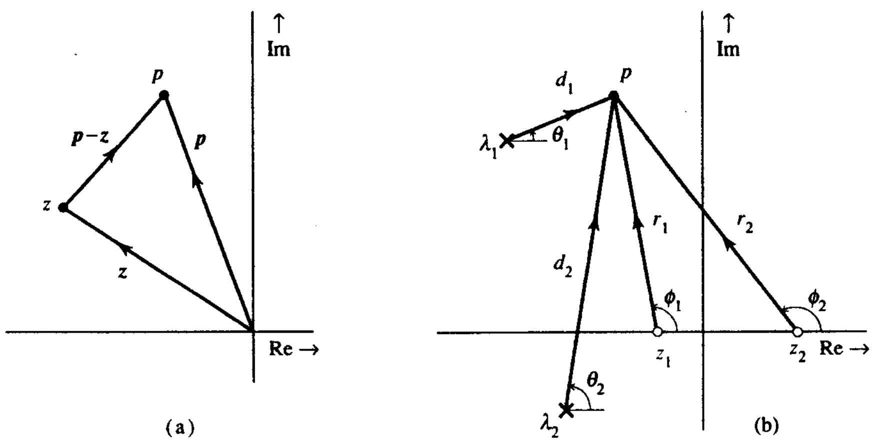


Fig. 4.38 (a)vector representation of complex numbers (b) vector representation of factors of $H(s)$.

4.7-1 Dependence of Frequency Response on Poles and Zeros of H(s)

Frequency response of a system is basically the information about the filtering capability of the system. We now examine the close connection that exists between the pole-zero locations of the system transfer function and its frequency response (or filtering characteristics). A system transfer function can be expressed as

$$H(s) = \frac{P(s)}{Q(s)} = b_n \frac{(s - z_1)(s - z_2)\cdots(s - z_n)}{(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)} \tag{4.78a}$$

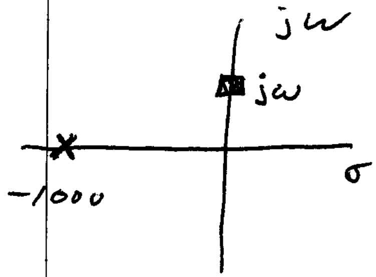
where $z_1, z_2, \ldots, z_n$ are the zeros of $H(s)$ and the characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the poles of $H(s)$. Now the value of the transfer function $H(s)$ at $s = p$ is

$$H(s)\big|_{s=p} = b_n \frac{(p - z_1)(p - z_2)\cdots(p - z_n)}{(p - \lambda_1)(p - \lambda_2)\cdots(p - \lambda_n)} \tag{4.78b}$$

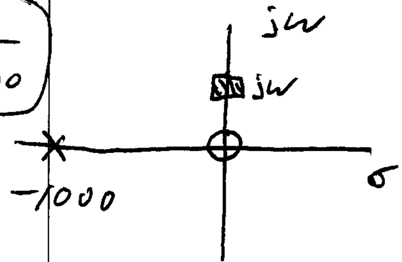
This equation consists of factors of the form $p - z_i$ and $p - \lambda_i$. The factor $p - z$ is a complex number represented by a vector drawn from point z to point p in the complex plane, as shown in Fig. 4.38a. The length of this line segment is $|p - z|$, the magnitude of $p - z$. The angle of this directed line segment (with horizontal axis) is $\angle(p - z)$. To compute $H(s)$ at $s = p$, we draw line segments from all poles and zeros of $H(s)$ to point p, as shown in Fig. 4.38b. The vector connecting a zero z_i to the point p is $p - z_i$. Let the length of this vector be r_i, and let its angle with the horizontal axis be ϕ_i. Then $p - z_i = r_i e^{j\phi_i}$. Similarly the vector connecting a pole λ_i to the point p is $p - \lambda_i = d_i e^{j\theta_i}$, where d_i and θ_i are the length and the angle (with the horizontal axis) respectively of the vector $p - \lambda_i$. Now from Eq. (4.78b) it follows that

$$\begin{aligned} H(s)\big|_{s=p} &= b_n \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_n e^{j\phi_n})}{(d_1 e^{j\theta 1})(d_2 e^{j\theta 2})\cdots(d_n e^{j\theta 1})} \\ &= b_n \frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} \, e^{j[(\phi_1 + \phi_2 + \cdots + \phi_n) - (\theta_1 + \theta_2 + \cdots + \theta_n)]} \end{aligned}$$

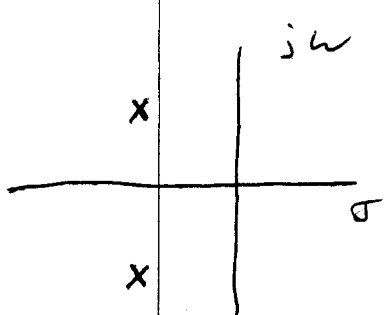
Examples: $H(s) = \dfrac{1000}{s+1000}$ Pole at $s = -1000$

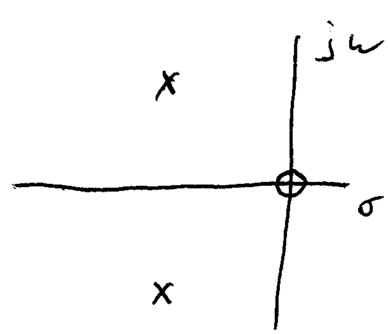


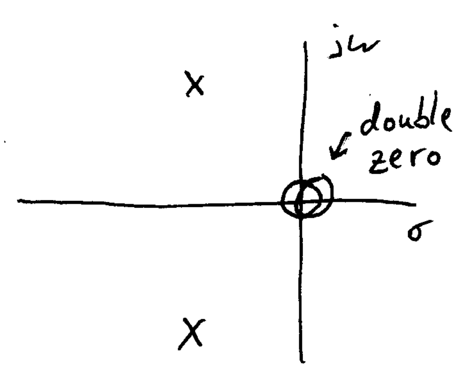
$\boxed{\dfrac{s}{s+1000}}$

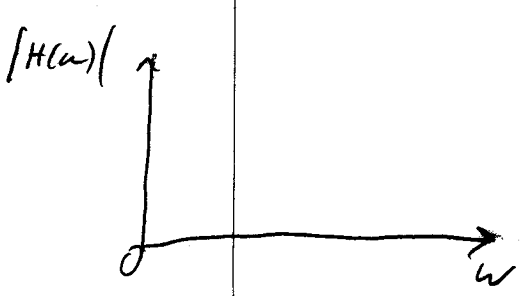


3 exercises:









Where should poles be placed for *ideal* low-pass filter?

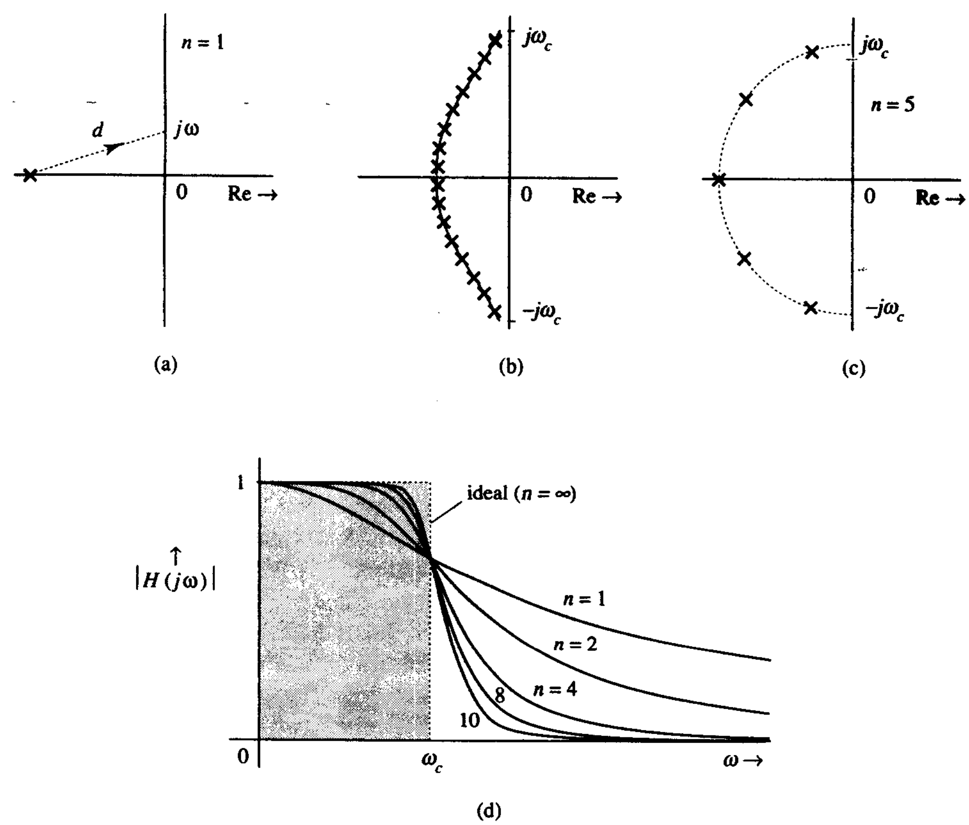


Fig. 4.40 Pole-zero configuration and the amplitude response of a a lowpass (Butterworth) filter.

"Butterworth": Place *finite* number of poles equally spaced around semi-circle.
$\Rightarrow$ Maximally flat passband.

"Chebycher": Place poles on semi-ellipse.
$\Rightarrow$ Faster cutoff, but "ripple" in pass-band.

Example: Notch filter