ELEC 470
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Review of Fourier Transform

The Fourier transform expresses a time signal as a sum of sinusoids with various frequencies, amplitudes, and phases.

<u>Def.</u>: For a time signal $g(t)$,

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt = \mathcal{F}[g(t)]$$

(Inverse F.T.)

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega = \mathcal{F}^{-1}[G(\omega)]$$

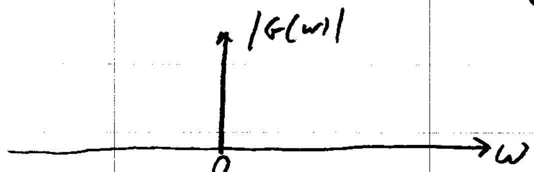
$$g(t) \Longleftrightarrow G(\omega) \qquad \text{F.T. "pair"}$$

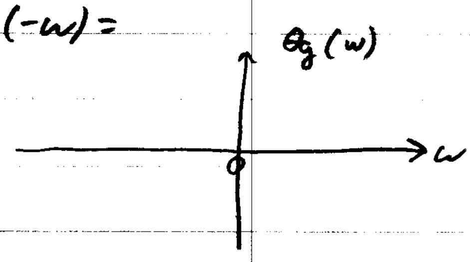
$G(\omega)$ is generally complex-valued even when $g(t)$ is real:

$$G(\omega) = \underbrace{|G(\omega)|}_{\text{Amplitude spectrum}}\, e^{j \underbrace{\theta_g(\omega)}_{\text{Phase spectrum } \theta_g(\omega)}}$$

Important <u>symmetry</u> of spectra when $g(t)$ is a <u>real</u>-valued signal:

$$G(-\omega) = G^*(\omega) \Rightarrow \begin{cases} |G(-\omega)| = \\ \theta_g(-\omega) = \end{cases}$$





So for <u>real</u> signals $g(t)$:

"Negative" frequencies in the spectrum can be constructed from the "positive" freq. components.
$\therefore$ Negative frequencies do not contain any information beyond what is contained in the positive freq. components.

(*) IMPORTANT FOR COMMUNICATIONS !

Recall what "negative frequency" is:

Euler's formula: $e^{j\phi} = \cos\phi + j\sin\phi$

$e^{-j\phi} =$

$\therefore \quad \cos\phi =$

$\sin\phi =$

<u>Ex</u>: Express a real cosine with complex exponentials:

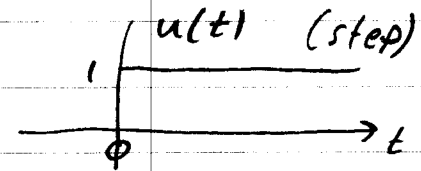
$3\cos\left(2\pi 100 t + \frac{\pi}{3}\right) =$

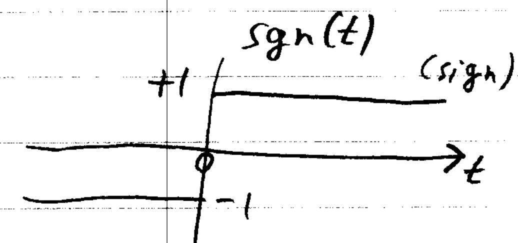
Table 3.1

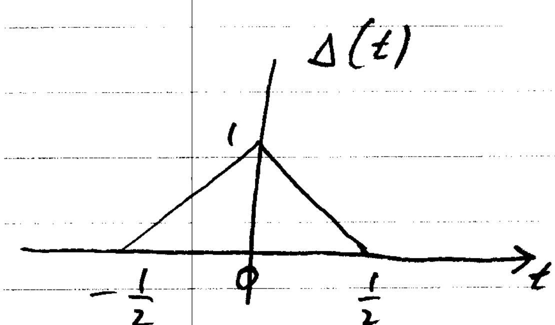
Short Table of Fourier Transforms

	$g(t)$	$G(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\,\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\,\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\text{sgn}\ t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\ u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\ u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\ u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\ u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\text{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\ \text{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\ \text{sinc}\ (Wt)$	$\text{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\ \text{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\ \text{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Important functions:







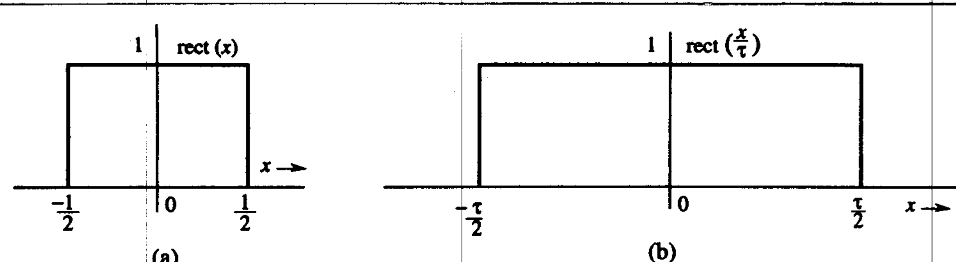


Figure 3.7 Gate pulse.

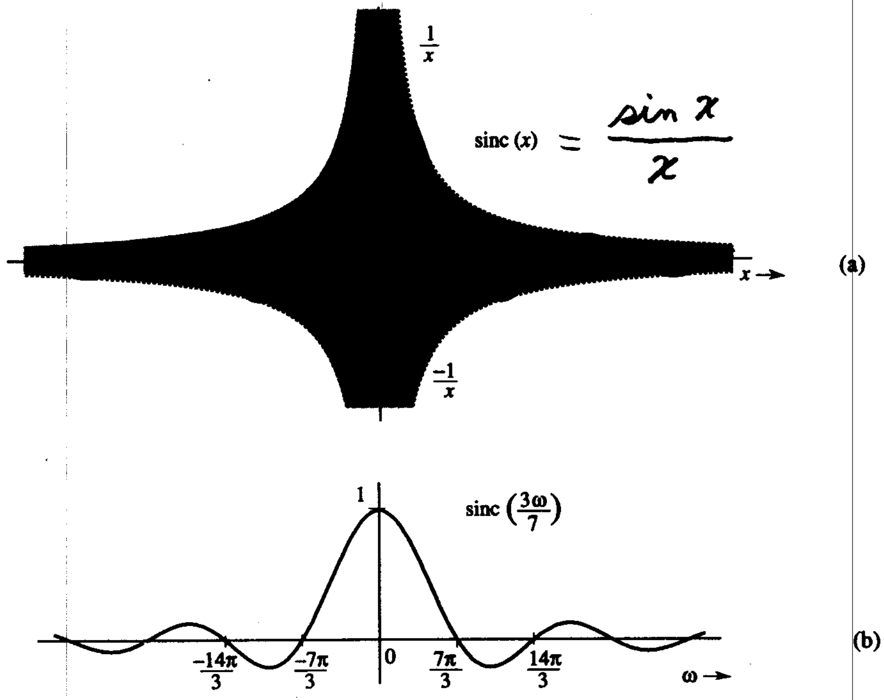


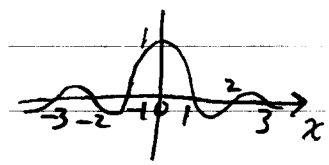
Figure 3.9 Sinc pulse.

$\angle \; \frac{3\omega}{7} = \pi \Rightarrow \omega = \frac{7\pi}{3}$

Note :

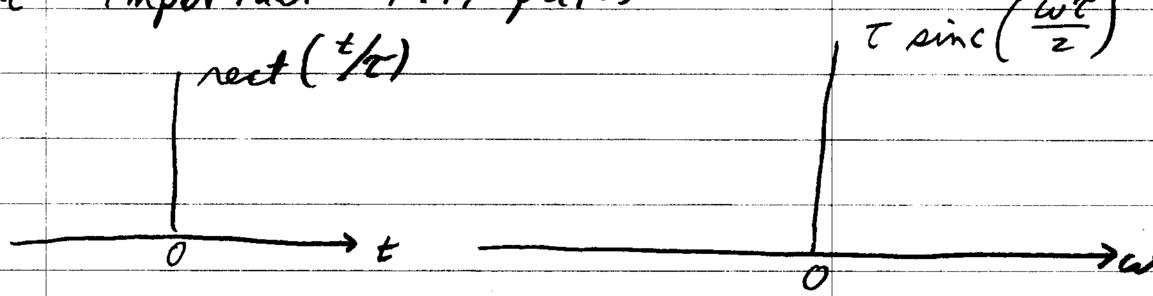
A more common def. is

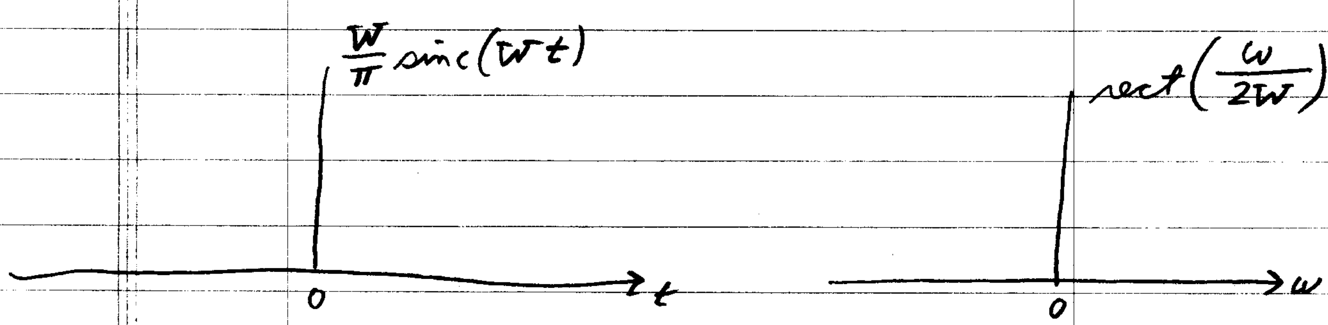
$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$



EXAMPLE 3.2 Find the Fourier transform of $g(t) = \text{rect}(t/\tau)$ (Fig. 3.10a).

Some important F.T. pairs :





Notes :

① Signal thinner in one domain ⟺ wider in other domain.

② What is the "bandwidth" of each signal above ?

Table 3.2
Fourier Transform Operations

Operation	$g(t)$	$G(\omega)$
Addition	$g_1(t) + g_2(t)$	$G_1(\omega) + G_2(\omega)$
Scalar multiplication	$kg(t)$	$kG(\omega)$
Symmetry	$G(t)$	$2\pi g(-\omega)$
Scaling	$g(at)$	$\frac{1}{\lvert a\rvert} G\left(\frac{\omega}{a}\right)$
Time shift	$g(t - t_0)$	$G(\omega)e^{-j\omega t_0}$
Frequency shift	$g(t)e^{j\omega_0 t}$	$G(\omega - \omega_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(\omega)G_2(\omega)$
Frequency convolution	$g_1(t)g_2(t)$	$\frac{1}{2\pi} G_1(\omega) * G_2(\omega)$
Time differentiation	$\frac{d^n g}{dt^n}$	$(j\omega)^n G(\omega)$
Time integration	$\int_{-\infty}^{t} g(x)\, dx$	$\frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$

$$g(t) \cdot \cos(\omega_0 t) \Leftrightarrow \frac{1}{2} G(\omega + \omega_0) + \frac{1}{2} G(\omega - \omega_0)$$

Rayleigh's energy thm.: $\int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$

Ex: Sketch Fourier transform (spectrum) of

$$g(t) = 3 \cos\left(2\pi 1000 t - \pi/4\right) - 1.7 \cos\left(2\pi 1500 t + \frac{4\pi}{3}\right)$$

Ex.: Cosine pulse, frequency ω_0, duration T:

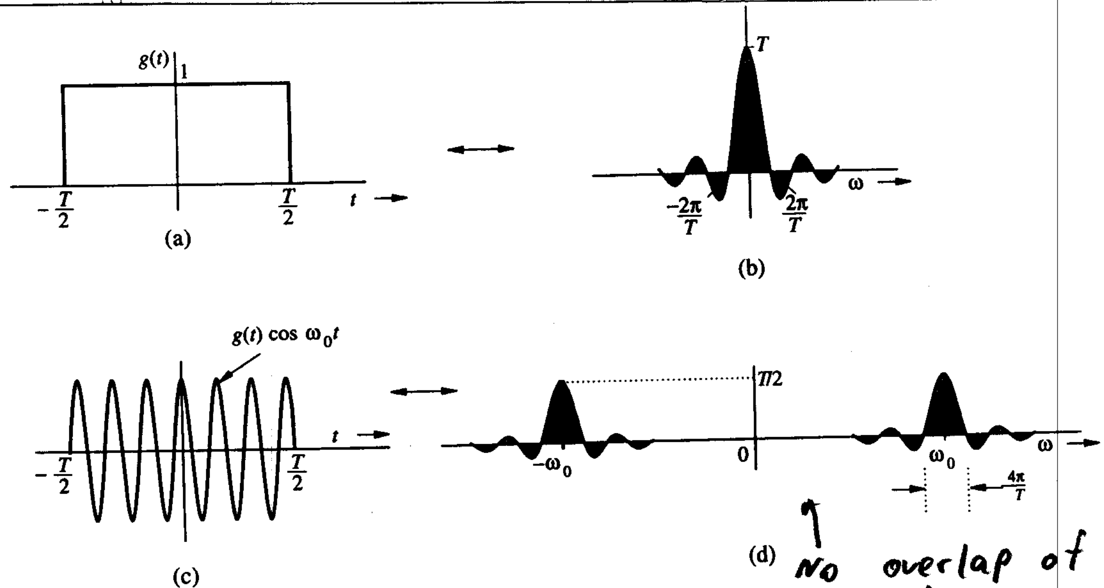


Figure 3.22 Example of spectral shifting by amplitude modulation.

No overlap of lobes if $\omega_0 \gg \frac{2\pi}{T}$ or $\omega_0 T \gg 2\pi$

Why ?