PHYS 221, Fall 2007
Lab 1

Introduction to Mathematica

Mathematica is a **very** powerful computer program. It can also be **very** frustrating to use when you are just learning, but learning how to use it is worth the trouble. In this lab you won't even scratch the surface of the capabilities of the program, but you will get a flavor of some of the things you can do with it. We will use *Mathematica* in subsequent labs in this course, and if you take other courses in physics at Bucknell you will probably see it again.

Try entering all the commands on the attached sheet. (This sheet is an actual *Mathematica* notebook.) Be very careful with all symbols - *Mathematica* is very picky about syntax. To enter a command, type it in, and then hit the `Enter` on the numeric keypad or the combination of `Shift` and `Enter` on the regular part of the keyboard. See if you can figure out what the command did, and then change something within the command to see if the output is what you expect.

By the end of this exercise you should be able to use *Mathematica* to

1. Do complicated arithmetic and algebra,
2. Define and plot functions,
3. Plot data,
4. Integrate functions,
5. Use "loops" to perform repeated calculations.

Here are a couple of things to keep in mind:

- All *Mathematica* functions begin with capital letters, and the argument of the function is always enclosed in single square brackets. **This is the only way that single square brackets are used.**
- In *Mathematica* curly brackets, `{` and `}`, are used for "lists." Lists can be literal lists of numbers, in which case they can represent vectors. They can be lists of lists of numbers, in which case they can represent matrices. But other things can go in lists too. For example, there can also be lists of things inside the arguments of functions (see the `Plot` function in the exercises). **This is the only way that curly brackets are used.**
- Elements of lists are referenced using double square brackets. For example, you can define `a` as a list of three numbers with the following command: `a = {5,10,11}`. The third element in the list of numbers is referenced as `a[[3]]`. **This is the only way that double square brackets are used.**
- Regular parentheses, `(` and `)`, are used to group terms in mathematical expressions, like `(a + (b+2)^2)^3`. **This is the only way that regular parentheses are used.**
- Adding a semicolon, `;`, after your input will suppress the printed output of your command.
- Comments that *Mathematica* will ignore can be entered between parentheses with stars in the following way: `(* Comment goes here *)`
- If you know the name of the *Mathematica* function you are trying to use, the quickest way to get online help is with a question mark. For example, if you want to use the `Plot` command, but you can't remember the syntax, enter the command `?Plot`. (If you want more detailed information enter the command `??Plot`, with two question marks.) After doing this I often click on the `More ...` link and scroll down to the `Further Examples` section of the Help page.

  The Mathematica Book (all 1488 pages) is also available as on-line help, but you have to learn how to navigate effectively.

Where to get more help

- **The Mathematica Book**, S. Wolfram, (Wolfram Media, 2003). This is the standard reference for version 5 of *Mathematica*. It has everything. Although it has lots of nice examples at the beginning to show you the capabilities of the program, it's not necessarily the easiest book to use to learn the basics.
- Web searches.
- **A Physicist's Guide to Mathematica**, P. T. Tam, (Academic).
- **Mathematica for Scientists and Engineers**, R. Gass, (Prentice-Hall)

Things to do

Examine the attached *Mathematica* "notebook" and try to figure out what the commands with stars beside them will do. (Come up with numbers, sketch graphs, etc., as necessary.) Then open up *Mathematica* and type in all of the commands in the "notebook" `intro.nb`, including the ones without the stars. See if you can figure out what the commands are doing. **After** they all make sense, complete the following exercises.

1. Find a numerical approximation for $\sqrt{270}$.
2. Define the function $f(x) = x^3 - 6x^2 + 5x - 10$.
3. Find a numerical value for $f(6781.2)$.
4. Plot the function $f(x)$. Select a scale that allows you to estimate visually the values of all roots of the equation $f(x) = 0$.
5. Find all real roots of the $f(x) = 0$ with values good to 5 significant figures.

6. Enter the following position *vs.* time data into a list named `xtData`. Plot the data with position on the vertical axis and time on the horizontal axis.

time (s)	position (m)
0	1
1	6
2	18
3	38
4	56
5	90

7. Plot the function $x(t) = 3t^2 + 2.2t + 1$ from $t = 0$ to $t = 5$.

8. Combine the plots from the previous two exercises onto a single graph.

9. Make a *Mathematica* list containing the first 40 positive integer powers of 2, *i.e.*, `{2,4,8,...}`. Extract from this list the 2^{35}, *i.e.*, the 35$^{\text{th}}$ element of the list.

10. Use a `For` loop to add up the cubes of the first 15 positive integers.

11. Look up information on how to use the *Mathematica* `Sum` command and repeat the previous calculation.

12. Assume the velocity of a particle as a function of time is given by $v(t) = 3+4t$. Assume that the initial position of the particle is given by $x(0) = -3.5$. Use the integrating capabilities of *Mathematica* to find an expression for the position as a function of time. (The necessary integration is easy to do by hand — check the result that *Mathematica* gives you.)

13. Redo homework probelm 2-10 using *Mathematica* commands discussed in this lab. Make a plot of the trajectory for the specific values $A = 1$ and $B = 1$.

14. Redo homework problem 2-12 using *Mathematica* commands discussed in this lab to help you do the "dirty work."

Notes To Instructors

- Several of the sample commands are *intended* to result in error messages, or output that is not comprehensible. For example:
  - `Solve[x^5 - x^2 + 6 == 0, x]` — gives some arcane information about rules to solve 5th order polynomials.
  - `v[[4]]` — asks for a part of a list that doesn't exist. Students should look at, and begin to recognize, the message
    `Part::partw: Part 4 of {5, 10, 11} does not exist.`
  - `Integrate[x^3, {x, a, b}]` — the first time this is called a is already defined as a List, so the output doesn't make sense.
  - `Plot[n*x + p, {x, 0, 4}];` — the first time this is called n and p are undefined, resulting in the message

    ```
    Plot::plnr: n x + p is not a machine-size real number
      at x = 1.6666666666666665‘*^-7
    ```
- Be clear about what you expect students to hand in. A few well chosen exercises with documentation is probably better than a printout of everything.
- In 2007 Sally and Marty both noticed less student familiarity with the concept of loops, iterators, etc. We surmise that programming languages like Basic are less prevalent in secondary schools these days. There is also confusion when you execute a loop and there is no output.

PHYS 221, Fall 2007
Lab 2

Analysis of motion using numerical integration I

Introduction

Consider the motion of a single particle of constant mass m in one dimension, say the vertical (y) dimension. Newton's second law,

$$F_{\text{net}} = ma, \tag{2.1}$$

is really a differential equation which must be solved to find the functions like $y(t)$, $v(t)$, and $v(y)$ which describe the position and velocity of the particle. To emphasize this characteristic of Newton's second law, perhaps it's better to rewrite Eq. (2.1) as a differential equation. There are several forms which this equation can take. It can be written

$$m\frac{d^2y}{dt^2} = F_{\text{net}}, \tag{2.2}$$

or equivalently

$$m\frac{dv}{dt} = F_{\text{net}}, \tag{2.3}$$

or, by using the chain rule,

$$m\frac{dv}{dy}\frac{dy}{dt} = F_{\text{net}}. \tag{2.4}$$

In simple cases, the net force F_{net} might be constant, but in general it might be a function of the particle's position or its velocity, and it might depend explicitly on time, *i.e.*, $F_{\text{net}} = F_{\text{net}}(y, v, t)$.)

The solution of a differential equation is a function, not a number. For example, the solution to Eq. (2.2) is the function $y(t)$, and the solution to Eq. (2.3) is the function $v(t)$.

For this lab you will start with an expression for the net force F_{net}, and you will solve for the functions $y(t)$ and $v(t)$. To do this, we will rewrite Newton's second law as the set of *coupled* differential equations

$$m\frac{dv}{dt} = F_{\text{net}}(y, v, t) \tag{2.5}$$

$$\frac{dy}{dt} = v. \tag{2.6}$$

Solving First-Order Differential Equations Numerically: Euler's Method

Sometimes you can solve the differential equations describing the motion analytically (as you will do in Exercises 2 and 3 below); this means that you will end up with *formulas* for $y(t)$ and $v(t)$. In other cases it will be difficult or impossible to find analytic functional forms for $y(t)$ or $v(t)$. In such cases you can get approximate solutions *numerically*. You are going to generate numerical solutions to differential equations step-by-step.

Starting from the definition of the derivative,

$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}, \tag{2.7}$$

it is straightforward to show that for small time steps Δt

$$v(t+\Delta t) \simeq v(t) + \frac{F_{\text{net}}}{m}\Delta t. \tag{2.8}$$

This says that if we know the velocity (v) and acceleration ($a = F_{\text{net}}/m$) at some time t, we can calculate an approximate expression for the velocity a little while later at time $t + \Delta t$.

In a similar manner, it is straightforward to show that

$$y(t+\Delta t) \simeq y(t) + v(t)\Delta t, \tag{2.9}$$

which says that we can calculate an approximate expression for the position at time $t + \Delta t$ if we know that position and velocity a little while earlier at time t.

Exercises

In the following exercises you will calculate the motion for two simple physical situations. You will start with a simple situation in which you will solve the problem three ways:

- you will derive exact analytic functions for $y(t)$ and $v(t)$,
- you will find approximate solutions for $y(t)$ and $v(t)$ at a number of discrete time points using a simple program that you write in *Mathematica*, and
- you will see how *Mathematica* find exact analytic functions for $y(t)$ and $v(t)$.

In more complex situations you may not be able to find analytic solutions to the equation of motion (Newton's second law), and you will have to rely on approximate numerical solutions.

Here's the physical situation: Consider a ball dropped from the top of a 50 m tower. To begin, let's assume that the ball is dropped at time $t = 0$, with velocity $v = 0$, and let's ignore air resistance and any other complicating factors that you can think of.

1. Draw a free-body diagram for the ball in its flight, and determine an expression for F_{net}. (This task is trivial in this example, but the procedure is one that we will use many times, so let's get in the habit of doing it.)
2. Find a function $v(t)$ that is a solution of Eq. (2.5). Choose any constants in your function so that the initial condition for the velocity at $t = 0$ is satisfied.
3. Find a function $y(t)$ that is a solution of Eq. (2.6) when you use the function $v(t)$ you just determined for the right-hand side of Eq. (2.6). Choose any constants in your function so that the initial condition for the position at $t = 0$ is satisfied.
4. Derive the approximations Eqs. (2.8) & (2.9) that are valid for small time intervals Δt, *i.e.,* show

$$v(t+\Delta t) \simeq v(t) + \frac{F_{\text{net}}}{m}\Delta t \tag{2.10}$$

$$y(t+\Delta t) \simeq y(t) + v(t)\Delta t. \tag{2.11}$$

5. Use the approximations you derived in the previous problem to fill in the blanks in the following table. (Use $g = 10\,\mathrm{m{\cdot}s^{-2}}$ for this exercise.)

t	$y(t)$	$v(t)$	F_{net}/m
0.0			
0.1			
0.2			
0.3			

6. Write a simple *Mathematica* program that uses the approximations you derived in the previous problem to find approximate values of $v(t)$ and $y(t)$ at 11 times from $t = 0$ to $t = 1\,\mathrm{s}$ that are separated by a time $\Delta t = 0.1\,\mathrm{s}$. Here are some suggestions:

   - Include a cell in which all physical constants are given symbols and values. Include things like the value of the acceleration, the initial time, the final time, the number of time points at which you want to calculate values, the initial position, the initial velocity.
   - Include a cell in which *Mathematica* lists are set up for the variables v, y, and t.
   - The calculations can be carried out in a *Mathematica* `For` function. In each pass, your program should calculate a new element in the velocity list `v[[i]]`, a new element in the position list `y[[i]]`, and a new element in the time list `t[[i]]`.

7. Make a graph of your approximate solution for $y(t)$ and $v(t)$. Use the *Mathematica* `ListPlot` function.

8. Make a graph of your analytic solution for $y(t)$ and $v(t)$. This will involve defining a function for $v(t)$ and $y(t)$. Use the *Mathematica* `Plot` function.

9. Combine the graphs for your approximate solution and your analytic solution for $v(t)$ using the *Mathematica* `Show` function.

10. Combine the graphs for your approximate solution and your analytic solution for $y(t)$ using the *Mathematica* `Show` function.

11. Investigate the quality of your approximation as you change the number of points you use in the time interval between 0 and 1 s.

12. Use the *Mathematica* `DSolve` function to have *Mathematica* find analytic solutions to Eqs. (2.5) and (2.6) that meet your initial conditions. You may need

some help with the syntax of the `DSolve` command, so don't be afraid to ask.

Now you will repeat the process, but this time you will incorporate air resistance into the problem. Air resistance contributes a force that depends on the velocity of the particle; a simple approximation for this force that works well for one-dimensional motion at low speeds for some particles in some media is

$$F_{\text{drag}} = -kv, \tag{2.12}$$

where k is a constant. We will use this expression for F_{drag} not because it is very realistic for all circumstances, but because it is an easy velocity-dependent drag force to work with, and any program that works for this force should be very easy to generalize for a more complicated velocity-dependent force. (There is a discussion of resistive forces, including numerical solutions of the equations of motion, on pp. 210–226 of your text, *Newtonian Mechanics*.) You will use exactly the same strategy you used in the previous exercise. The only difference is that the force on the ball will not be constant. This should involve you having to make only a few changes to the notebook you wrote for the previous exercise.

13. Draw a free-body diagram for the ball in its flight, and determine an expression for the net force F_{net}.

14. Determine the terminal velocity of the ball.

15. Use the approximations you derived in Exercise 4 to fill in the blanks in the following table. (Use $g = 10\,\text{m}{\cdot}\text{s}^{-2}$, $m = 1.0\,\text{kg}$, and $k = 0.5\,\text{N}{\cdot}\text{s}{\cdot}\text{m}^{-1}$. This value for k is not a value based on reality — it's a value based on getting nice numerical results in this exercise.)

t	$y(t)$	$v(t)$	$a(t) = F_{\text{net}}(t)/m$
0.0			
0.1			
0.2			

16. Write a simple *Mathematica* program that uses the approximations you derived in Exercise 4 to find approximate values of $v(t)$ and $y(t)$ at 11 times from $t = 0$ to $t = 1s$ that are separated by a time interval of $\Delta t = 0.1s$. (This should be a simple modification of the program you wrote for the case of free-fall with no air resistance.)

17. Make a graph of your approximate solution for $y(t)$ and $v(t)$. Use the *Mathematica* `ListPlot` function.

18. Combine the graphs for your approximate solution for $v(t)$ with air resistance and your analytic solution for $v(t)$ with no air resistance using the *Mathematica* `Show` function.

19. Combine the graphs for your approximate solution for $y(t)$ with air resistance and your analytic solution for $y(t)$ with no air resistance using the *Mathematica* `Show` function.

20. Investigate the quality of your approximation as you change the number of points you use in the time interval between 0 and $1s$. Do you get the expected terminal velocity?

21. Use the *Mathematica* `DSolve` function to have *Mathematica* find analytic solutions to Eqs. (2.6) and 2.3 that meet your initial conditions. You may need some help with the syntax of the `DSolve` command, so don't be afraid to ask.

What to include in your notebook

Your notebook should include answers to all the numbered exercises in the previous section. This lab is different than a true experimental lab in that you may want to work some things out on scratch paper, and have your notebook be an organized record of your answers. Some of your answers will be in the form of *Mathematica* output that you can tape into your notebook. All output **must** be documented - you should insert comments into your *Mathematica* code.

Notes to Instructors Based on 1998 Experience

These exercises were done on the computers in Olin 271. In the Fall of 98 these were still 486's running Windows 95, and we used the X-Win emulator to connect to charcoal.

I had the students start by giving the students a 2 page *Mathematica* notebook printout with lots of simple *Mathematica* input commands on it: some some with arithmetic operations, some with algebraic operations, some with Lists, Vectors, and Matrices, some with plotting and graphics, and some with simple programming ("DO loops"). I had the students type in the commands as written, examine the output, and try to figure out what the program was doing. All of the commands necessary for the lab were on the sheet of course.

I then had them work through this sheet. The whole idea of programming was new to some of the students. For example, the translation of Euler's equations to statements like `v[[i]] = v[[i-1]] + a[[i-1]] dt` was not easy for everybody. Also, the difference between `=`, and assignment operator, and `==` as used in `Solve`, etc. was new to some.

All students should use $g = 10$. Makes checking of tables a heck of a lot easier.

Not all students "get" the idea that they should check the computer output against the data in the table they calculate by hand.

When doing this from X-Win (or Suns in the Physics Department) you should telnet to castor or pollux. (**Not** charcoal.)

PHYS 221, Fall 2007
Lab 3

Analysis of motion II: Comparison of numerical approximation with experimental measurements

Introduction

In this two-week lab experiment, we will analyze the motion of real balls falling under the influence of gravity, and include the effect of air resistance. There are two parts of this lab: the collection of data, and the analysis of this data using *Mathematica* notebooks that are very similar to those you developed in the last lab exercise. In the experimental section, you will drop balls from the balconies overlooking the foyer between the Biology and Chemistry buildings, and measure the time it takes for the balls to reach the floor. In the analysis section, you will build on your work from the previous exercise using a realistic expression for the drag force, and calculate the position of the ball as a function of time, $y(t)$. The two sections of this exercise may be done in either order.

Experiment

1. Go to the foyer between Biology and Chemistry and drop balls from the two balconies. Your goal is to obtain the best possible measurement of the time it takes for a variety of different balls to hit the ground. You will be provided

with measuring tapes and stop watches. Try various methods and decide upon the best technique.

Be sure to practice and repeat your measurements enough times so that *i*) you become proficient at making consistent measurements, and *ii*) you get a feel for the range of values that are reasonable given the equipment you have. Determine the uncertainty due to random errors in your timing measurements. (Each member of a group should try to make some timing measurements.) You should get "good" data on several balls.

2. Determine your group's best value for the time for each ball and each height, **including a justified estimate of your uncertainty**. You should strive for a precision of better than 5%.

3. Measure any physical properties of the ball that you will need to compare your data with your theoretical prediction.

4. Make (and name) a list in your *Mathematica* program that contains your best values for the two pairs of distance and time that you measured.

5. Plot your data for each ball together with a theoretical prediction for height as a function of time, assuming that the balls move under the influence of gravity with no air resistance. Comment on this comparison.

Analysis

In this part of this lab you will continue your analysis of the motion of a single particle falling under the influence of gravity, and you will include a realistic expression for air resistance. You will build on your work from Laboratory 2 and use your program to calculate something you can actually measure. The goal is to determine theoretically the position of the ball as a function of time, $y(t)$. This will allow you to predict the time it takes for a ball to fall to the floor from each of the levels of balconies in the foyer between the Biology and Chemistry buildings, and see if your predicted times are consistent with measured times.

Air resistance contributes a force that depends on the velocity of the particle. On p. 213 of your text *Newtonian Mechanics*, French discusses a "Detailed Analysis of Resisted Motion." He asserts that for spherical objects the size of ping pong ball falling in air, the *magnitude* of the drag force can be written

$$F_{\text{drag}} = C_2 r^2 v^2, \tag{3.1}$$

where r is the radius of the ball, v is the speed of the ball, and C_2 is a constant whose value is 0.87 kg/m^3. You will have to make slight modifications to the program you wrote in Lab #2 to incorporate this form of a velocity-dependent force.

As before, the differential equations we are trying to solve are

$$\frac{dv}{dt} = \frac{F_{\text{net}}}{m} \tag{3.2}$$

and

$$v = \frac{dy}{dt}, \tag{3.3}$$

and we will be using the approximations

$$v(t+\Delta t) \simeq v(t) + \frac{F}{m}\Delta t, \tag{3.4}$$
$$y(t+\Delta t) \simeq y(t) + v(t)\Delta t. \tag{3.5}$$

Note that for our specific case, $F_{\text{net}} = F_{\text{net}}(v)$, i.e., the net force depends on the ball's speed.

Exercises

1. Draw a force diagram for the ball in its flight, and determine an expression for $F_{\text{net}}/m = a(t, x, v)$ that you can use in the right side of Eq. (3.2).
2. Use your results from the previous problem to determine the *terminal velocity* of the ball.
3. For numerical convenience, use the values $C_2 = 1$, $m = 1$, $y(0) = 0$, $r = 1$, and $g = 10$ to fill in the blanks in the following table:

t	$y(t)$	$v(t)$	$a(t) = F_{\text{net}}(t, y, v)/m$
0.0 s			
0.1 s			
0.2 s			
0.3 s			

4. Write a simple *Mathematica* program to find approximate values of $v(t)$ and $y(t)$ and $F(t)/m = a(t, x, v)$ at 11 times from $t = 0$ to $t = 1$ that are separated by a time $\Delta t = 0.1$ s. You should base this program on your successful program from Lab #2. Be sure to check the values calculated by your program against your entries in the table above. Here are some suggestions for your program:

- Include a cell in which all physical constants are given symbols and values. Include things like the value of the initial time, the final time, the number of time points at which you want to calculate values, the initial position, the initial velocity, and the value of the constant C_2 in the drag force.

- Define a *Mathematica* function that calculates $a = F_{\text{net}}/m$ as a function of velocity.

- Include a cell in which *Mathematica* lists are set up for the variables v, y, a, and t.

- The calculations can be carried out in a *Mathematica* `For` function. In each pass your program should calculate a new element in the velocity list `v[[i]]`, a new element in the position list `y[[i]]`, a new element in the acceleration list `a[[i]]`, and a new element in the time list `t[[i]]`.

5. When your program is working and giving the same answers as your entries in the table, change the parameters to reflect the physical situation you are studying, *i.e.*, one of the real balls in free fall. (If you haven't measured the properties of the balls yet, do so now.) Make a graph of your approximate solution for $y(t)$ and $v(t)$. Use the *Mathematica* `ListPlot` function.

6. If the results of your program look reasonable, increase the number of points you calculate between $t = 0$ and $t = 1$ (or equivalently decrease your time step Δt) until your solution doesn't change with smaller values of Δt.

7. Use the following checks to make sure your program is giving you reasonable results:

   (a) Set the constant $C_2 = 0$ in your expression for the drag force and make a graph that shows the solution from your program **and** the analytic solution you derived in Lab #2 for the case of no air resistance. Comment on the results of this exercise.

   (b) Make a graph that shows your numerical solution for $C_2 = 1$ **and** the analytic solution you derived in Lab #2 for the case of no air resistance. Does your solution deviate from the analytic solution in a "reasonable" way? (You may have to adjust some scales to combine graphs successfully. If you don't know how to do this, get some help.)

   (c) Does your solution reach a terminal velocity? Make whatever adjustments are necessary so that you can check whether your program gives you the terminal velocity you expect from the calculation you did in Exercise 2 above.

Combining experiment and theory

1. Adjust the parameters in your *Mathematica* notebook so that your results will correspond to the physical situation that you have measured. Now create plots of

    (a) your prediction for $y(t)$,

    (b) your actual data for individual balls, and

    (c) $y(t)$ in the case of no air resistance.

    Finally, combine all your plots on a single graph.

2. So what about Galileo? Seriously, what can you say about his purported experiments in which he dropped balls from the leaning tower of Pisa and showed that they all fell at the same rate? Do all balls hit the ground at the same time? Was Galileo a liar? Was he wrong? Or can you justify his conclusions in some limit?

Optional extra exercise

If you have time, extend your program so that it can handle projectile motion in two dimensions. For uniformity, let's all start with the case of $v_0 = 10$ m/s directed at an angle of $\theta = 30°$ above the horizontal.

What to include in your notebook

Your notebook should include answers to all the numbered exercises in the previous section. Some of your answers will be in the form of *Mathematica* output that you can tape into your notebook. All output **must** be documented - you can do this neatly by hand, or you can insert comments into your *Mathematica* code. This notebook must also include your experimental data, including a description of your experimental technique.

PHYS 221, Fall 2007
Lab 4

Functional Relationships from Simple Oscillator Data

Introduction

Suppose you have data on two related quantities, and you want to see if you can determine a functional form for the relationship. One way is to "look for" specific kinds of relationships. For example, if you suspect that two variables have a linear relationship you can simply plot them to see if you are right.

If you suspect that there might be a power-law relationship, *i.e.*,

$$g(v) = av^b, \tag{4.1}$$

where a and b are constants, it will be a little more difficult to see if you are right by simply plotting g *vs* v, and it wouldn't be easy to get a value for a or b from the graph. But there are some straightforward techniques you can use to make things easier. For instance, you can take the logarithm of both sides of Eq. (4.1) to get

$$\log(g) = \log(a) + b\log(v). \tag{4.2}$$

Now if you plot $\log(g)$ *vs.* $\log(v)$ you will get a linear relationship **if** your hunch about the functional form of the relationship is correct.

If you suspect that the relationship has the form

$$g(v) = a\exp(bv) = ae^{bv}, \tag{4.3}$$

where once again a and b are constants, you can play the same sort of game.

Procedure

1. Measure the spring constant k of a spring. (Remember, $|F_{\text{spring}}| = kx$.) As you know, the best techniques will involve multiple measurements. You should use the *Mathematica* `Fit` function which has the following format:

   $$\texttt{Fit[data,\{1,x\},x]} \tag{4.4}$$

   where `data` is a set of x-y pairs. You may have entered your data as a set of x-y pairs, e.g., `data = {{x1,y1},{x2,y2},...}` or you may entered the data as separate lists `x={x1,x2,...}` and `y={y1,y2,...}`, in which case you can convert it into pairs with something like
   `data = Transpose[{x,y}]`.

2. Measure the period of oscillations of your spring when a variety of masses is hung from the spring.

3. Make a graph of period *vs* mass. Do you get a linear relationship?

4. Make a graph that helps you determine whether your data is described by a power law relationship like that in Eq. (4.1).

5. Discuss with your instructor the effect of small masses like those in the spring.

6. Determine a functional form that describes the relationship between period and mass. Use the fitting function in Eq. (4.4) with your data to determine the value of the power law.

7. Now measure the amplitude as a function of time for a slowly decaying oscillation.

8. Determine the functional relationship between amplitude and time.

Notes to Instructors

General

In 1999 Sally and Marty did this for the first time. Oscillations hadn't been covered in class, so this was purely a "discover the functional relationship" lab. Not one student knew what the functional form of period vs mass "should" be, and that made the lab great. (We didn't push them to dredge up their memories of PHYS 211, or equivalent, but we did inquire about it.) We left the interpretation of the intercepts in the log-log and semi-log plots to future labs. None of this was review to the students. In 2007, this was done at the same time as oscillations in class, and it works well with this sequence too.

Spring Constant

We used the air track apparatus to do this, but without using the air. We simply secured one of the carts to the track with a rubber band, attached a spring to the cart, attached a string to the other end of the spring, hung the string over one of the pulleys at the end of the track, and hung small weights on a hangar attached to the string. There isn't actually any need to use the air track: hanging the spring vertically from any fixed mount would work.

The spring constant data isn't really used in this lab (after it is measured, but this data is used in the *next* lab. (It's a good idea to label the springs so data can be used later.) But it's good to include it here for a couple of reasons:

1. it's a nice linear fit,

2. it's good to discuss "best" ways to determine k from a data set (many students want to find a whole bunch of k's from pairs of data points, and then average them), and

3. 3) it's a nice introduction to the *Mathematica* `Fit` function.

The natural tendency is to plot F *vs.* x. M.L. talked to individual groups about errors, plotting, least-squares fitting, etc., but didn't make a big deal about it at this time.

Old PHYS 221 problem comes up: Is it better to time 50 periods once, or 10 periods 5 times?

Period *vs* mass

NOTE: You could also measure frequency *vs.* mass.

- Advantage: The graph is more obviously non-linear than graph of period *vs.* mass. This makes "conversion" to linear form more dramatic. (It's also easier to use with "bad" data.)
- Disadvantage: The graph is more obviously non-linear than graph of period *vs.* mass. If you want to make the point that a graph can appear at first to be pretty linear, but closer examination can reveal small systematic deviations from a straight line, then the period data is better to use.

It's extremely important to start with masses that are as small as possible. Otherwise the square root function looks too linear. You are looking at $m = m_0 + \Delta m$, and if $\Delta m/m \ll 1$, then

$$T = c\sqrt{m} \simeq c\sqrt{m_0}\left(1 + \frac{1}{2}\frac{\Delta m}{m_0}\right).$$

M.L. used masses in the range of "empty hangar" up to 50 g. Curvature is slight, but readily apparent with good data.

When doing the log-log plot of the period/mass data, the slope is sensitive to the small amount of mass in the spring (and the pulley if you're doing this on the air tracks). Just including the mass of the weights leads to a slope of less than 0.5. You can make things "better" by pulling out the formula on p. 61 of French's *Vibration and Waves* that includes mass, i.e.,

$$\omega^2 = \frac{k}{m + M/3}.$$

Amplitude *vs* time

For this we did use the air tracks with air. We set up a cart attached to a spring on both ends, and the springs were attached to fixed points. The oscillation decays over a long enough time that the students can easily note the position of the maximum extent of the oscillation at a given time. Data collected in this experiment really looks nonlinear if you record over several half-lives, and looks bad in a log-log plot, but looks very linear (especially for the first few half-lives) in the semi-log plot.

PHYS 221, Fall 2007
Lab 5

Rotational Oscillations I

Introduction

In the laboratory exercise *Functional Relationships* you investigated the dependence of the period of a simple harmonic oscillator on the mass of the oscillating object. For rotational oscillators the period depends not only on the magnitude of the mass, but also on the distribution of the mass relative to the center of rotation. You will use the same techniques you used in the *Functional Relationships* experiment to deduce the functional form of this dependence.

From the results of the experiment *Functional Relationships* you know that the period T of a simple harmonic oscillation is related to the mass m in the following way:

$$T = c\sqrt{m}, \tag{5.1}$$

where c is a constant. (From the theory of simple harmonic oscillation, it's straightforward to figure out a formula for the constant, c.)

In this lab you will investigate the period of a rotational oscillation as a function of the position of the mass on an extended object. The masses will sit on a rotating plate, and the plate has its "own" period of oscillation. You should experiment with masses totaling about 900 g, placing them at various positions on the plate, and seeing how the position affects the period. From your qualitative experiments, and your results from the *Functional Relationships* experiment, it should be plausible that we can write the period of oscillation as

$$T = c_1\sqrt{I_0 + mr^{\alpha}}, \tag{5.2}$$

where I_0 is a property of the plate alone, and the exponent α is unknown. In this lab, you will determine the constant α.

To facilitate your work, it is convenient to rewrite Eq. (5.2) as

$$\left(T^2 - c_2 I_0\right) = c_2 m r^\alpha, \tag{5.3}$$

where $c_2 = c_1^2$.

Procedure

1. Pick some fixed amount of mass m, say 900 g, and place it as close to the center of the plate as possible (so that $r \simeq 0$) and measure the period of oscillation.
2. Distribute the same amount of total weight symmetrically around the plate at some known radius r and re-measure the period.
3. Repeat your measurement of the period for at least 6 different radii for the mass.
4. Analyze your data to see if it has the functional form of Eq. (5.3). If it does, determine the constants α and c_2.

Notes to Instructors

This experiment uses the trifilar pendulums that will be used later to measure moments of inertia. Marty began with a simple demonstration that the position of the mass on the oscillating plate makes a dramatic difference in the period. Then he tried to motivate Eq. (5.2) as a plausible functional form.

This lab involves no new techniques. All *Mathematica* commands have been used before; the log-log plotting idea was used previously (although in this lab we're taking log's of a manipulated quantity).

Results are pretty good. The error introduced in not being able to place all of the mass at $r = 0$ isn't terribly significant. If you stack a few heavy cylindrical weights on top of each other, the effective moment of inertia is on the order of $0.5 \times m \times (1\text{ cm})^2$, which is about 1% of the moment of inertia when the masses are distributed out at $r = 10$ cm.

We did this lab in conjunction with a wrap-up of Lab 3. See handout for Lab 3 wrap-up.

Motivation of Lab:
Sally began discussion of this lab at a time when the class was covering rotational motion. She began by reminding people of the results from the previous lab, the period of a pendulum varies according to the square root of the mass. It is then a natural extension to discuss the trifilar pendulum as the rotation of an extended object (the plate). As such, it is straightforward to motivate that, by complete analogy, it can expected that

$$T \propto \sqrt{I_0}$$

where I_0 is the moment of inertia of the plate (making up the pendulum).

The goal of the experiment is framed as: Given that the period of the pendulum goes as the square root of the moment of inertia, we can use this fact to probe the functional relationship of the moment of inertia. To do this, masses will be added to the plate of the trifilar pendulum; they will be added at a fixed radius. The goal is then to determine the power law of the added moment of inertia. Specifically, what is α if $I = mr^\alpha$?

This is done by beginning with the assumption

$$T \propto \sqrt{I_0 + I_{\text{added}}}$$

where $I_{\text{added}} = mr^\alpha$.

Then the task becomes, "What must be plotted so that α can be extracted from the data?" This leads naturally to the expression $(T^2 - c_2 I_0) = c_2 m r^\alpha$.

Aside: A value of I_0 must be determined which carries through all the calculations in this experiment. After some discussion on the board, it becomes apparent that I_0 can be determined by measuring the period when the radius $r \to 0$. This can be accomplished in two ways which appear equivalent to students at first: 1) take off the six masses (constituting the added masses to the plate) and then measuring the period, or 2) put all six masses as close to the center of the plate ($r = 0$) and measure the period. Are these methods equivalent? I had each group measure the period using both methods. It turns out that there is a difference of almost a factor of two. I then tell them to think about the 'system' and which case keeps the system as constant as possible with only a rearrangement of the masses. Clearly, the tension in the strings of the trifilar pendulum is different when the masses are removed. Keeping the masses on the plate and moving them to the center does not introduce any changes to the system.

PHYS 221, Fall 2007
Lab 6

Driven, Damped, Harmonic Oscillations McAllister Apparatus

Experiment

The McAllister Apparatus is a realization of a driven, damped mechanical oscillator. In this lab we will make measurements on a damped, driven mechanical oscillator to determine how the amplitude of oscillation and phase of oscillation depends on the frequency of the driving force.

1. In order to ensure that your equipment is set up correctly, perform the following steps:

   (a) Turn on the power supply voltage to +12 Volts.

   (b) On the McAllister control box, set switch S1 in the "up" position and switch S2 in the "down" position.

   (c) Turn on the oscilloscope to view the pulses that are sent to the digital *stepper* motor. Set the oscilloscope to the following ranges: volts/div = 2.00 V and sec/div = 1.00 ms. You should be able to measure the pulse period P from the oscilloscope display. The black knob on the control box adjusts the pulse period.

   (d) For each pulse sent to the stepper motor, the motor turns by 1.8°. Therefore from measuring the pulse period P, you should be able to determine the oscillation period T of the rotor and therefore determine the driving frequency. Use a stop watch to measure the oscillation period for one setting, and confirm that this is the same as you get from the pulse period P.

2. Make a qualitative sketch of the amplitude of the oscillation *vs.* drive frequency, and phase of the oscillation (relative to the drive) *vs.* drive frequency.
3. Take careful data of amplitude *vs.* period and use this data to make a plot of amplitude *vs.* angular frequency ω.
4. Measure the spring constant k, and the mass m of the oscillator.
5. Make a rough measurement of the damping constant by observing the decay of free oscillations of your oscillator.
6. Compare your results with the theory you derive in the following section.

Complex Number Exercises

For these exercises, consider that following complex numbers:

$$\begin{aligned} z_1 &= 3 - 4i \\ z_2 &= -i \\ z_3 &= 2e^{-i\pi/3} \\ z_4 &= 6e^{i\pi/6} \end{aligned}$$

All complex numbers can be written in several forms. For this exercise, let's label two of these forms as follows:

Form A: $z = a + ib$, where a and b are real numbers, and

Form B: $z = Re^{i\theta}$, where R and θ are real numbers.

Remember that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{6.1}$$

1. For complex numbers z_1 through z_4:
   (a) Calculate the magnitude $|z|$ of the number.
   (b) If the number is given in **Form A**, identify the real numbers a and b. If the number is given in **Form B**, identify the real numbers R and θ.

(c) If the number is given in **Form A**, change it to **Form B** and identify the real numbers a and b. If the number is given in **Form B**, change it to **Form A** and identify the real numbers R and θ.

(d) Sketch a phasor representation of the number in the complex plane.

2. Use Eq. (6.1) to write an expression for $e^{-i\theta}$ in terms of sines and cosines.

3. Combine your result from the previous problem with Eq. (6.1) to write expressions for $\cos\theta$ and $\sin\theta$ in terms of complex exponentials.

Solution of the Equation of Motion

Consider Newton's second law for a driven damped oscillator:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F_{\text{net}} \\ &= F_{\text{spring}} + F_{\text{damp}} + F_{\text{drive}}. \end{aligned} \tag{6.2}$$

If we assume a sinusoidal driving term, a damping that is proportional to velocity, and a linear spring, this equation can be written

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 \cos\omega t. \tag{6.3}$$

Dividing through by m gives:

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma\frac{dx}{dt} + \frac{F_0}{m}\cos\omega t, \tag{6.4}$$

where $\omega_0^2 \equiv k/m$ and $\gamma \equiv b/m$. Now let's let turn this into a differential equation for a *complex* function of time z, whose real part will be the actual physical displacement x. The complex form of Eq. (6.4) is

$$\frac{d^2z}{dt^2} = -\omega_0^2 z - \gamma\frac{dz}{dt} + \frac{F_0}{m}e^{i\omega t}, \tag{6.5}$$

From your experimentation with the McAllister apparatus you should see that the steady state motion is sinusoidal at the *drive* frequency ω, although the oscillation may not be in phase with the drive. This suggests that we look for solutions of the form

$$z = Ae^{i(\omega t-\delta)}. \tag{6.6}$$

The constant A will be the amplitude of the motion, and the constant δ will give the phase *lag* of the displacement with respect to the driving force. (If the displacement is in phase with the driving force, then $\delta = 0$.)

4. Plug the assumed form of the displacement from Eq. (6.6) into the differential equation of motion, Eq. (6.5). Take all time derivatives, solve for A, and find the magnitude $|A|$. You should find an amplitude that depends on the drive frequency:

$$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2\right]^{1/2}} \tag{6.7}$$

5. Compare your results with your theoretical predictions.
6. If you have time, examine the phase lag of your oscillator as a function of the driving frequency ω.

Notes to Instructors

There seems to be a systematic that we haven't accounted for. Measuring the spring constant and mass, or equivalently the period when the mass is hanging in air, gives an ω_0 which is large compared to the position of the measured resonance peak. I think this may be due to the effect of the buoyant force changing with y, but I'm not sure. Perhaps make hangars with "all" mass at the bottom. (ML, 11/2001)

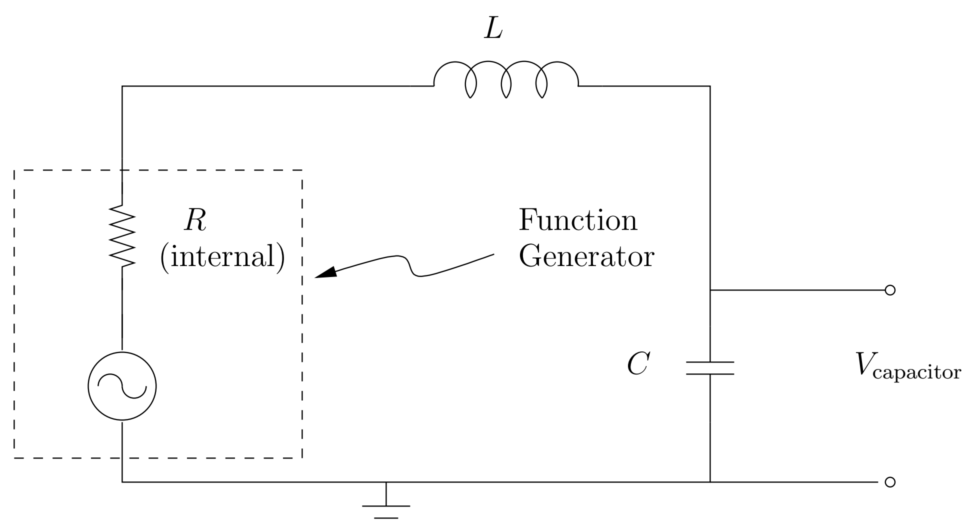
2007 Update: There was a real and significant systematic problem. Dave Schoepf and M.L. recall that in the "old days" we used masses with very different vanes: We both remember thin brass (?) vanes on thin rods. In recent years we have been using 1/4 inch threaded rod with thick Lucite crosses at the bottom. Dave Vayda doesn't agree remember the former apparatus the way I do, but I'm pretty sure he's the one who constructed the current Lucite crosses. In 2006 I had two students do a project to improve the results. They made the bobber out of a coat hanger with clay mass on the coat hanger ABOVE the surface of the water. (They also embedded some extra steel balls in the clay for extra mass.) This greatly reduces the damping, and the majority of the mass is out of the water at all times. They got very good results, but because the Q is so large it was difficult to take data with our existing stepper motor apparatus. It's not only tough to map out the resonance curve because it's so narrow, but the mass jumps out of the liquid if your don't have just the right mass, drive amplitude, etc.

In 2007 I had everybody use thrown together bobs made out of thin aluminum rod with shim stock vanes duct-taped to the bottom. I also duct-taped some standard disk weights near the top of the aluminum rods. In spite of the crude apparatus we got much, much better quantitative agreement than in recent years. Is this the result of reducing the effect of time-varying buoyant forces, or the result of going to a regime of damping proportional to v rather than v^2? I'm not sure, but I think it's the latter. (Shear forces vs. end-on forces?) The 2006 students did a little bit of numerical investigation of v^2 damping, but it wasn't very complete or documented.

PHYS 221, Fall 2007
Lab 7

Driven, Damped, Electrical Oscillations

This lab is the electrical analog of the driven, damped, mechanical oscillations lab that you recently completed. In place of the McAllister machine you will construct a simple circuit consisting of a function generator (voltage source), a resistor, a capacitor, and an inductor (coil). The circuit is illustrated below:



We will get an "equation of motion" for the charge q on the capacitor by using Kirchoff's Law: the sum of the voltage drops around a loop must be zero. If we call the current in our circuit i, the voltage drop across the resistor is

$$V_{\text{resistor}} = iR, \tag{7.1}$$

the voltage drop across the capacitor is

$$V_{\text{capacitor}} = \frac{q}{C}, \tag{7.2}$$

and the voltage drop across the inductor is

$$V_{\text{inductor}} = L\frac{di}{dt}. \tag{7.3}$$

Adding up the voltage drops around the circuit gives

$$\begin{aligned} 0 &= V_{\text{generator}} - V_{\text{resistor}} - V_{\text{inductor}} - V_{\text{capacitor}} \\ &= V_0 \cos\omega t - iR - L\frac{di}{dt} - \frac{q}{C} \end{aligned} \tag{7.4}$$

Using the fact that $i = dq/dt$, and rearranging, gives

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 \cos\omega t. \tag{7.5}$$

This should look familiar in form! The equation of motion of the driven damped *mechanical* oscillator that you have been working with recently is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos\omega t \tag{7.6}$$

You should argue by analogy to prove the following relations giving the amplitude and the phase (relative to the driving voltage) of the voltage across the capacitor:

$$V_{\text{capacitor}} = \frac{V_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}, \tag{7.7}$$

and

$$\tan(\delta) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}, \tag{7.8}$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \text{and} \qquad \gamma = \frac{R}{L}. \tag{7.9}$$

Procedure

1. Set up the circuit illustrated in the figure. Use the oscilloscope to observe simultaneously the voltage across the terminals of the function generator and the voltage drop across the capacitor.

2. Set the function generator to give sinusoidal oscillations. Observe the amplitude and phase of the voltage across the capacitor as you vary the frequency of the oscillator. Find the resonant frequency that gives the maximum voltage. Before taking any careful data make a qualitative sketch of the amplitude of the voltage across the capacitor as a function of the frequency of the function generator. Make a second *qualitative* sketch of the phase lag of the voltage across the capacitor (relative to the drive voltage from the function generator) as a function of frequency.

3. Make careful measurements of the amplitude and phase of the voltage across the capacitor at about 20 frequencies near the resonance frequency. Choose your frequencies so that they fall on both sides of resonance.

4. Plot your amplitude *vs* frequency data with a computer. Also plot the theoretical function giving the amplitude as a function of frequency. Adjust your damping constant to give the best fit. Is the damping constant what you "expect" it to be?

5. Plot your phase *vs* frequency data with a computer. Also plot the theoretical function giving the phase as a function of frequency.

6. Now set up the function generator to give you a relatively low frequency square pulse. You should see damped *free* oscillations of your electrical system that are analogous to those you would get if you gave a mechanical oscillator a single kick. Measure the frequency of the oscillations you see, and make an approximation of the damping constant γ from the half-life of the decay of the oscillations. The frequency you measure should be close to the frequency you predict from theory, and close to the frequency at which you got the largest $V_{\text{capacitor}}$ when the circuit was driven with a sine wave. Verify these assertions.

Notes to Instructors

1. There is always the perennial choice of which voltage to measure. In PHYS 235 we often measure voltage across the resistor because the current is the easiest thing to derive: $i = v/Z$. As of 2007 this is written to measure the current across the capacitor. This is the closest analog to the physical displacement measured in the McAllister machine.

2. In 2007 M.L. began to realize how bad standard inductors can be. The faded green potted inductors are the highest Q inductors we have. The inductors in the silver "cans" have about the same inductance, but much lower Q. The low Q affects the result very significantly if you're looking for quantitative agreement. Modeling including the effect of the imperfect inductors (with frequency-dependent properties) continues.

3. We have hand-held inductance meters now that effectively replace the old GR Bridge.

PHYS 221, Fall 2007
Lab 8

How Good is Your Fit? An Introduction to Residuals

Introduction

In many scientific experiments you have a mathematical model that expresses a relationship between physical quantities, and the goal of an experiment is to determine some parameters in the model. For example, in Lab 4, *Functional Relationships from Simple Oscillator Data*, we assumed that there was a linear relationship between the extension of a spring and the force exerted on the spring:

$$|F_{\text{spring}}| = k|x|. \tag{8.1}$$

In this lab you will take the same kind of data you did in Lab 4 and measure the restoring force for a different kind of oscillator, and then you will look more closely at the relationship between linear extension and restoring force.

Residuals

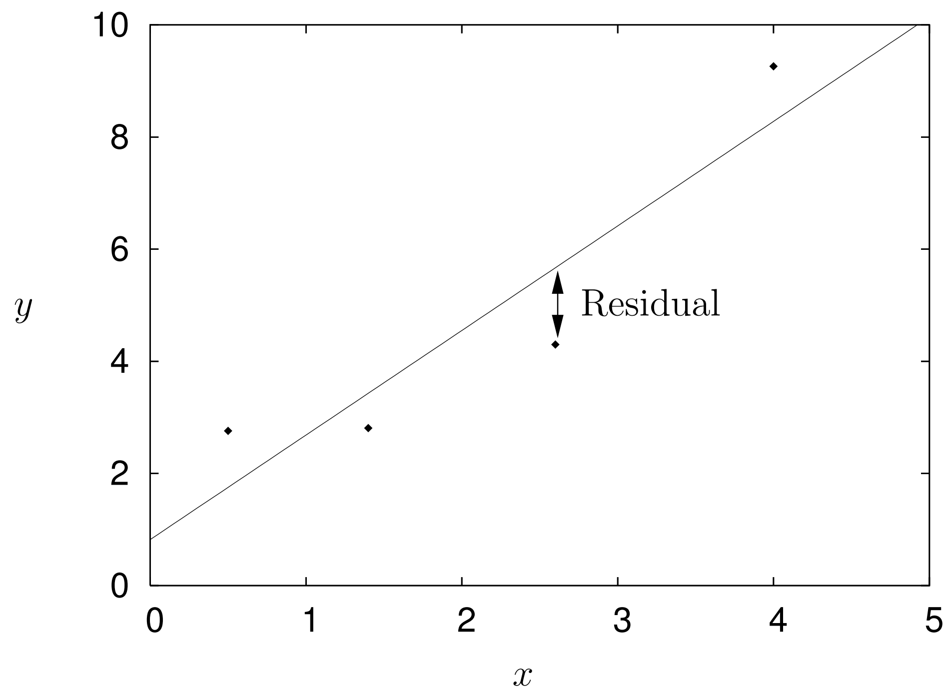
In assessing whether data actually fits an assumed model, it's useful to examine *residuals*. Residuals are the difference between the actual data and the values predicted by the model. Let's imagine that you recorded the following data for two experimental variables x_i and y_i:

i	x_i	y_i	$y(x_i)$	Residual
1	0.5	2.76	1.76	1.00
2	1.4	2.81		
3	2.6	4.30		
4	4.0	9.26		

The x_i's are assumed to have negligible uncertainty, so they are plotted along the horizontal axis. A linear least-squares fit to the data gives the function

$$f(x) = 1.87x + 0.82,$$

which is plotted along with the data in the graph below.



The *residuals* are the vertical differences between the data and the line that is presumed to fit the data. A mathematical representation of the i^{th} residual r_i is

$$r_i = y_i - f(x_i).$$

A graph of the residuals *vs.* x should show a random scatter of points about 0, and the magnitudes of the residuals should be consistent with the estimated uncertainties in the measured values y_i.

Procedure

1. Fill in the empty boxes in the sample data table above.
2. For about 15 masses between 0 and 150 g. Carefully measure the vertical position of the suspended mass for about 15 masses between 0 and 150 g.
3. Estimate your uncertainty in your measurements of vertical position.
4. In your data set there the masses are known to high precision, and your measurements of position have some accompanying uncertainty. Therefore make a graph of position *vs.* mass (position on the vertical axis; mass on the horizontal axis).
5. Fit your data to a straight line, and plot the resulting line on the same graph with your data. Does the fit look good?
6. Make a graph of your residuals. What does this graph say about the quality of your linear fit?
7. Fit your data to a quadratic ($a + bm + cm^2$), and plot the resulting parabola on the same graph with your data. Does the fit look good?
8. Make a graph of your residuals. What does this graph say about the quality of your quadratic fit?
9. Fit your data to a cubic ($a+bm+cm^2+cm^3$), and plot the resulting parabola on the same graph with your data. Does the fit look good?
10. Make a graph of your residuals. What does this graph say about the quality of your cubic fit?
11. Discuss your results with your instructor.

Notes to Instructors

This was a new lab in 2006 thrown together by M.L. A couple students really liked it, and there was some evidence that it had some value in the 2007 version of PHYS 329.

This is a simple non-linear oscillator constructed from the big gyroscopes. The central bar of the gyroscope is clamped in a horizontal position, and things are arranged so the big disk hangs out over the edge of the table. A couple of standard disk masses totaling about 100 g are taped to the big disk very near the rim. A piece of string is draped over the top of the disk; one end is taped down, and the free end is draped over the top of the disk and allowed to hang vertically. Masses are hung from the string and displacements are measured. This is very much like the measurement of the spring constant in the *Functional Relationships* lab. (For best results, very fine string or thread should be used.) The data looks pretty linear on a gross scale, but residuals show that it deviates systematically from a straight line. The linear fits have a very high value of R^2, but this is a good chance to talk about why that's not a particularly good measure here.

PHYS 221, Fall 2007
Lab 9

Rotational Oscillations II

Introduction

In *Rotational Oscillations I* you investigated how the period of rotational oscillations of an extended object depended on the position of the mass. You measured the period of oscillation of a "trifilar pendulum" consisting of a metal disk suspended by three strings, with some "extra" mass m placed on the disk. You found that

$$T = c_1\sqrt{I_0 + mr^2}, \tag{9.1}$$

where I_0 is a property of the plate, r is the distance of the "extra" mass from the axis of rotation.

The *moment of inertia*, I, of an extended object (like the plate itself) is just the sum of lots of terms like the mr^2 in Eq. (9.1), with a term for every little piece of mass that comprises the object:

$$I \equiv \sum_i r_i^2\,\Delta m_i, \tag{9.2}$$

which in the limit of a continuous mass distribution becomes

$$I = \int r^2\,dm. \tag{9.3}$$

The equation of motion for rotations about a fixed axis is a simple generalization of Newton's second law. For a mass m on a linear spring with spring constant k we have

$$m\frac{d^2x}{dt^2} = -kx, \tag{9.4}$$

which has the solution

$$x(t) = A \sin\left(\sqrt{\frac{k}{m}}t + \phi\right). \tag{9.5}$$

For rotations, we are concerned with angular displacements θ instead of linear displacements x, and we replace mass m with moment of inertia I, and force F with torque τ. For rotations of a rigid body about a fixed axis the equation of motion becomes

$$I\frac{d^2\theta}{dt^2} = \tau_{\text{net}}. \tag{9.6}$$

If the torque is linear in θ, and acts to return the body to its equilibrium position, then this equation becomes

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta, \tag{9.7}$$

where κ is just the proportionality constant between torque and angular displacement. Comparing this equation to Eq. (9.4) we see that the solution is

$$\theta(t) = A \sin\left(\sqrt{\frac{\kappa}{I}}\,t + \phi\right). \tag{9.8}$$

In this lab, you will determine an expression for the torsion constant κ in terms of the physical properties of your apparatus, you will measure the period of the "empty" trifilar pendulum, and you will combine these results to determine a value for I_0, the moment of inertia of the disk. You will compare this to a theoretical calculation of the moment of inertia of the disk.

Procedure

1. For small angular displacements θ, the restoring torque provided by the strings on the disk is approximately proportional to θ. Consider how the strings exert torques on the plate, and show that

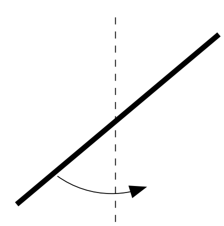
$$\kappa = \frac{MgR^2}{L},$$

   where M is the total mass suspended by the strings, L is the length of the strings, g is the acceleration due to gravity. and R is the perpendicular distance from the center of rotation to the point of application of the force resulting in the torque.

2. Measure the period of oscillation of your plate. From this measurement and your expression for κ, determine an experimental value for the moment of inertia of the plate.

3. Use Eq. (9.3) to calculate a theoretical formula giving the moment of inertia of your metal plate. To get a numerical value for the moment of inertia, you will have to determine the mass of the plate and measure its physical dimensions. Compare this value to that obtained in part 2.

4. Determine a theoretical formula for the moment of inertia of a thin rod about its center. Measure the physical properties of one of the rods in the lab and determine a theoretical value for the moment of inertia of the rod.



5. Measure the period of oscillation of the combined plate plus rod. From this measurement determine an experimental value for the moment of inertia of the rod. Compare your experimental result with the value obtained in part 4

6. Calculate the moment of inertia of a solid sphere.

7. Measure the period of oscillation of the combined plate plus sphere. From this measurement determine an experimental value for the moment of inertia of the sphere. Compare your experimental result with the value obtained in part 6.

Notes to Instructors

1. .

PHYS 221, Fall 2007
Lab 10

Gyroscopes and Rotational Motion

Introduction

During this lab, you will become familiar with the basic principles underlying the motion of the gyroscope, and also investigate some other situations involving rotational motion.

The Gyroscope

The gyroscope is an example of the application of Newton's second law for rotation,

$$\frac{d\mathbf{L}_c}{dt} = \sum_i \boldsymbol{\tau}_i, \tag{10.1}$$

where $\mathbf{L}_c$ is the angular momentum vector about the center of mass, and the $\boldsymbol{\tau}_i$'s are the (vector) torques about the center of mass.

The components of the gyroscope you will be using consist of a flywheel (which can spin), a support base, and two movable counterweights attached to the rod on which the flywheel spins.

1. Without spinning the flywheel, use the counterweights to balance the gyroscope. Draw the free–body diagram associated with the balanced gyroscope; be sure to label all the forces present.
2. Where is the center of mass of the system in the balanced condition? Write down an expression for the center of mass of the system in terms of m_1, m_2, m_3, r_1, r_2, and r_3. (I suggest letting r_1, r_2, r_3 be positive, and make all signs explicit.)
3. What is the net torque on the system in the balanced condition? Write down an expression for the net torque (about the center of mass) acting on the system in terms of m_1, m_2, m_3, r_1, r_2, and r_3.
4. While holding the gyroscope in the balanced conditions, set the flywheel spinning and then release it. Describe what happens. Is this consistent with Newton's second law?
5. Stop the flywheel and return the gyroscope to its balanced, equilibrium position. Now move the heavy counterweight (m_2) away from the equilibrium position by an amount Δr.
   (a) Determine the new position of the center of mass in terms of m_1, m_2, m_3, r_1, r_2, and r_3, and Δr.
   (b) Show that the magnitude of the new net torque about the center of mass is $\tau = m_2 g \Delta r$.
   (c) Determine the direction of the new net torque.
6. Predict the magnitude of the angular velocity of the precession in terms of measurable quantities.
7. Start the flywheel spinning, and "play" with the apparatus.
8. With the flywheel spinning, get the gyroscope precessing evenly. Measure ω_{spin} with the strobe, and $\Omega_{\text{precession}}$ with a stop watch.
9. Make any other physical measurements necessary to compare your predicted precession frequency with your observations.

Notes to Instructors

PHYS 221, Fall 2007
Lab 11

Coupled Oscillations and Normal Modes

Introduction

In previous classes and labs you have studied the motion of single, isolated oscillators. In this lab you are going to investigate *coupled* oscillators. Coupled oscillators can have very complex motions, but there exist states of the system, called *normal modes*, in which the motion is quite simple. A normal mode of a system of oscillators is a state in which all parts of the system oscillate at the same frequency.

In this experiment you will investigate the oscillations of rods hung as pendula from a common steel band. The pendula are coupled because they "feel" the presence of their nearest neighbors via the twisting of the steel band.

Theory

Each swinging rod obeys Newton's second law in the form

$$\sum_i \tau_i = I_{\mathrm{rod}} \frac{d^2\theta}{dt^2}, \tag{11.1}$$

where θ is the angular displacement of the rod from equilibrium (*i.e.*, vertical) and $\sum \tau_i$ is the net torque on the rod. The net torque acting on a given rod is the sum of three parts:

1. the gravitational torque,
2. a torque caused by any twist in the segment of the band immediately to the left of the rod, and
3. a torque caused by any twist in the segment of the band immediately to the right of the rod.

The torque produced by the steel band is proportional to the angle of the twist of the band in a given section divided by the length of that section of the band:

$$|\tau_{\text{band}}| = \kappa \frac{|\Delta\theta|}{s}, \tag{11.2}$$

where $\Delta\theta$ is the twist angle of a section of band of length s, and κ is the torsion constant of the steel band that you will determine experimentally.

For two pendula suspended symmetrically with respect to the center of the band, there are two simple normal modes:

$$\textbf{Mode I:} \qquad \theta_1(t) = \theta_2(t) \tag{11.3}$$

$$\textbf{Mode II:} \qquad \theta_1(t) = -\theta_2(t) \tag{11.4}$$

For three pendula suspended at $D/4$, $D/2$, and $3D/4$ there are three normal modes:

$$\textbf{Mode I:} \qquad \theta_1(t) = -\theta_3(t); \quad \theta_2(t) = 0 \tag{11.5}$$

$$\textbf{Mode II:} \qquad \theta_1(t) = \theta_3(t) = \frac{\theta_2(t)}{\sqrt{2}} \tag{11.6}$$

$$\textbf{Mode III:} \qquad \theta_1(t) = \theta_3(t) = -\frac{\theta_2(t)}{\sqrt{2}} \tag{11.7}$$

Procedure

Theory

1. Show that the moment of inertia of a rod of mass M and length L which rotates around one end is given by

$$I_{\text{rod}} = \frac{1}{3}ML^2.$$

2. Show that the gravitational torque acting on a single rod which is displaced from equilibrium by an angle θ is given by

$$\tau_{\text{grav}} = -\frac{1}{2}MgL\sin\theta \simeq -\frac{1}{2}MgL\theta.$$

3. Show that the torque due to the steel band on a single rod suspended from the center of the band is given by

$$\tau_{\text{band}} = -\frac{4\kappa}{D}\theta,$$

   where D is the total length of the band.

4. Write down the equation of motion for a single rod placed at the center of the band, a distance $D/2$ from either end. Find a theoretical expression for the angular frequency of oscillation.

5. Find an algebraic expression for the torsion constant κ in terms of the quantities M, I, L, D, and T (the period of oscillation of the single bar).

6. Consider the case of two rods suspended at $2D/5$ and $3D/5$. Write down an equation of motion for each of the two rods.

7. For the case of two bars, use the information given in Eq. (11.3) to find a theoretical expression for the frequency (and period) of normal mode I.

8. For the case of two bars, use the information given in Eq. (11.4) to find a theoretical expression for the frequency (and period) of normal mode II.

9. Consider the case of three rods suspended at $D/4$, $D/2$, $3D/4$. Write down an equation of motion for one of the three rods.

10. Find a theoretical expression for one of the normal mode frequencies for the three-rod case.

Experiment

1. Measure the mass of the rods, the length of the rods, and the length of the steel band.

2. Measure the period of the single-rod system. From this period and your expression for the frequency, calculate the torsion constant κ.

3. Measure the periods of both normal modes of the two-rod system.

4. Measure the periods of all three normal modes of the three-rod system.

5. Compare your results for the normal mode frequencies with your theoretical predictions.

Notes to Instructors

- The "small" setups are probably better than the large, floppy, "old" setups. The good ones are made of two-by-fours, and use the shorter rods.
- The brass rods are stored in the green cabinet with drawers at the back of the electronics lab. (One of the lower drawers.)
- Rocking of the apparatus can be a problem. (The modes of oscillation include more than just the rods!) Make sure to stabilize the apparatus with wooden wedges from the drawer with the rods.
- Be careful about confusion between angular frequency ω and the "other" $\omega = \dot{\theta}$.
- This worked well in 2001 because it was preceded by a lecture on the three-spring–two-mass problem in class. We discussed the equations of motion for the individual masses, the idea of normal modes, and then we "guessed" the motion, and solved for the normal mode frequencies. The context of the lab problems is different enough that it wasn't too repetitive. I also previewed some stuff about rotational dynamics that we will be getting to shortly.
- Be sure to tighten the blade for each set of equipment. This makes a remarkable difference to the success of the lab.
- It's important to keep amplitude of oscillations *very* small. There are three reasons:

  1. usual small angle approximation,
  2. this keeps excitation of rocking modes of entire apparatus small, and
  3. the band doesn't loosen with small amplitude oscillations.