Topic 5

Representation of Sinusoidal Functions with Complex Numbers and Phasors

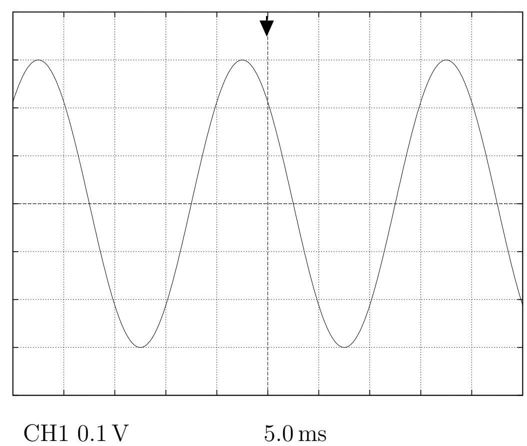
Material discussed 1/29/19 – 1/31/19

5.1 Introduction

It is very useful to be able to represent sinusoidal functions in several ways, some of which exploit the relationship between sines and cosines and complex exponentials. The goal is to be able to convert easily between four representations of a sinusoidal function like those displayed in Fig. 5.1. If you see one representation, the others should be at your fingertips.

All of the representations in Fig. 5.1 refer to the same electrical oscillation: a cosine function with an amplitude of $0.3\,\mathrm{V}$, a period $T = 20\,\mathrm{ms}$, a frequency $f = 1/T = 50\,\mathrm{Hz}$, and angular frequency $\omega = 2\pi f = 100\pi \simeq 314.16\,\mathrm{s}^{-1}$, and a phase shift of $\phi = 45^\circ = \pi/4\,\mathrm{rad} \simeq 0.79\,\mathrm{rad}$. Representation A is a graphical representation like you might see on the face of an oscilloscope; Representation B is the standard algebraic representation of a cosine function with the appropriate amplitude, angular frequency, and phase. Representations C and D may not be as familiar to you — they display algebraic and graphical representations of complex numbers that are used to represent the oscillation.

Representation A



Representation B

$$v(t) = 0.3\cos(100\pi t + \pi/4)$$

Representation C

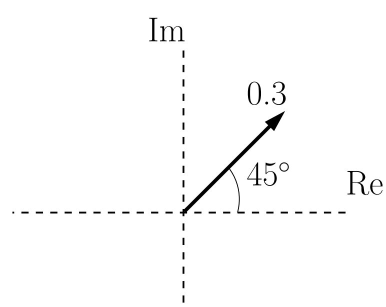
$$\tilde{v}(t) = 0.3\, e^{j(100\pi t + \pi/4)}$$

or

$$v(t) = 0.3\angle 45^\circ$$

(frequency implied)

Representation D



(frequency implied)

Figure 5.1: Four equivalent representations of the same sinusoidal function that might appear on an oscilloscope screen as in Representation A.

5.2 Review: Complex Numbers and the Complex Plane

The imaginary unit j is defined such that

$$j^2 = -1. \tag{5.1}$$

(In electronics and electrical engineering we use j instead of i to avoid confusion with current, which is often denoted by i.)

Complex numbers are numbers that are the linear combination of an imaginary part and a real part:

$$z = a + jb, \tag{5.2}$$

where a and b are both real numbers. We say that the real part of the complex number z is a (or $\mathrm{Re}(z) = a$) and the imaginary part is b (or $\mathrm{Im}(z) = b$). Complex numbers can be plotted in the *complex plane*, where the real part of the number is plotted along the horizontal axis, and the imaginary part of the number is plotted along the vertical axis. For example, the complex number $z_1 = 4 + 3j$, with a real part of 4 and an imaginary part of 3, and the complex number $z_2 = 4 - 3j$, with a real part of 4 and an imaginary part of -3, are plotted in Fig. 5.2.

The *complex conjugate* of a complex number is formed by taking all j's and turning them into $-j$'s, e.g., if $z = a + jb$, then the complex conjugate, denoted with an asterisk, is $z^* = a - jb$. In the example illustrated in Fig. 5.2, z_2 is the complex conjugate of z_1.

The *absolute value* of a complex number, $|z|$, often called the *modulus* is equal to the distance from the origin to the point as plotted in the complex plane. For a complex number $z = a + jb$, the absolute magnitude is

$$|z| = \sqrt{a^2 + b^2}.$$

It is useful to note the relationship

$$|z| = \sqrt{z \times z^*}.$$

5.3 Review: The complex exponential

Euler's relations,

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \text{and} \qquad e^{-j\theta} = \cos\theta - j\sin\theta \tag{5.3}$$

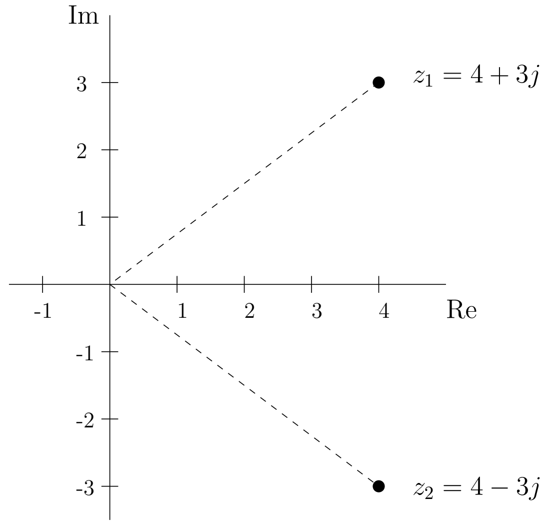


Figure 5.2: Plotting of complex numbers in the complex plane.

can be inverted to give

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \tag{5.4}$$

These relations suggest an alternative way to write complex numbers. For example, consider the quantity $Re^{j\theta}$. Using the relations above we find that

$$Re^{j\theta} = R(\cos\theta + j\sin\theta) = R\cos\theta + jR\sin\theta. \tag{5.5}$$

This is just a complex number with a real part of $R\cos\theta$ and and imaginary part of $R\sin\theta$. Writing complex numbers in the form of complex exponentials encourages the geometric interpretation of complex numbers in the complex plane as is illustrated in Fig. 5.3 for the general complex number $z = R\,e^{j\theta} = a + jb$. Relationships between the quantities a, b, R, and θ are fundamentally geometric in nature, and they are analogous to the relationships between the components of a vector and the vector's magnitude and direction.

5.4 Arithmetic of Complex Numbers

Consider two complex numbers,

$$z_1 = a_1 + jb_1 = R_1\,e^{j\theta_1} \qquad \text{and} \qquad z_2 = a_2 + jb_2 = R_2\,e^{j\theta_2}. \tag{5.6}$$

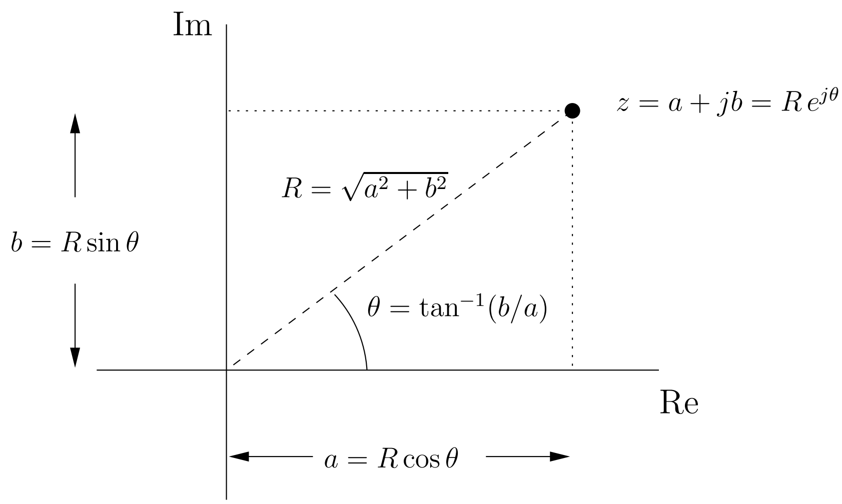


Figure 5.3: Geometric interpretation of complex exponential representation of complex numbers.

Addition: To add complex numbers simply add the real parts and imaginary parts separately. This is easiest in the "$a + jb$" representation:

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2). \tag{5.7}$$

Multiplication: Complex numbers are easiest to multiply in the "$R\,e^{j\theta}$" representation:

$$z_1 \times z_2 = R_1 R_2 e^{j(\theta_1+\theta_2)}, \tag{5.8}$$

although it's not too bad as

$$\begin{aligned} z_1 \times z_2 &= (a_1 + jb_1)(a_2 + jb_2) \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1) \end{aligned} \tag{5.9}$$

Here's an example of division:

$$\begin{aligned} \frac{3 + j4}{2 - 2j} &= \frac{5\,e^{j\tan^{-1}(4/3)}}{\sqrt{8}\,e^{j\tan^{-1}(1)}} \\ &= \frac{5}{2\sqrt{2}}\,e^{j(\tan^{-1}(4/3) - \tan^{-1}(1))} \\ &\simeq 1.77 e^{0.142j} \end{aligned} \tag{5.10}$$

5.5 AC Signals and Complex Numbers

Consider a sinusoidal signal

$$v_1(t) = R_1 \cos\omega t. \tag{5.11}$$

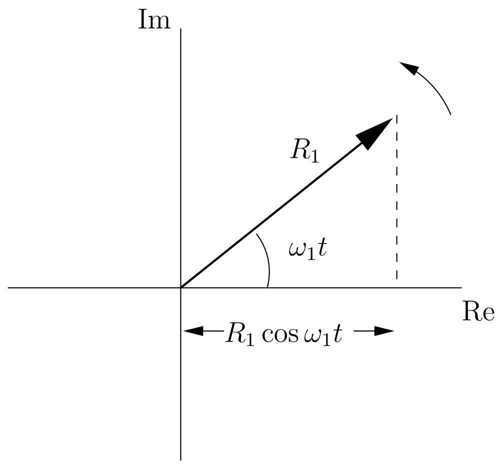


Figure 5.4: Rotating phasor whose real part corresponds to a real ac signal.

This signal is the *real part* of the complex function

$$\tilde{v}_1 = R_1 \, e^{j\omega t} = R_1 \cos \omega t + j R_1 \sin \omega t. \tag{5.12}$$

The complex quantity $\tilde{v}_1$ can be represented by a rotating *phasor* in the complex plane, and the real signal is the projection of the rotating phasor on the real (horizontal) axis. The phasor rotates counterclockwise as the angle $\omega_1 t$ increases with time as is illustrated in Fig. 5.4.

Complex representation of sinusoidal signals becomes very useful when you want to add signals that are out of phase with each other. (This is the kind of thing you need do with Kirchoff's loop rule in AC circuits.) For example, consider the addition of the signal $v_1(t)$ to a second signal

$$v_2(t) = R_2 \cos(\omega t + \phi_2). \tag{5.13}$$

This signal is the real part of

$$\tilde{v}_2 = R_2 \, e^{j(\omega t + \phi_2)} = \left(R_2 \, e^{j\phi_2} \right) e^{j\omega t} \tag{5.14}$$

At time $t = 0$ the phasors representing the two signals are illustrated in Fig. 5.5. The complex signal $\tilde{v}_1$ is entirely real at this time, and the signal $\tilde{v}_2$ is oriented at an angle ϕ_2 above the real axis. The sum of the two phasors representing $\tilde{v}_1$ and $\tilde{v}_2$ is also shown in the figure. As time increases from $t = 0$ all the phasors will rotate counterclockwise an angular frequency ω, but the lengths and relative orientations of the phasors will remain the same — the phasor representing $\tilde{v}_2$ will always be *ahead* of the phasor representing $\tilde{v}_1$ by an angle ϕ_2, and the phasor representing the sum of the two signals will always be ahead of $\tilde{v}_1$ by the same angle α (and behind

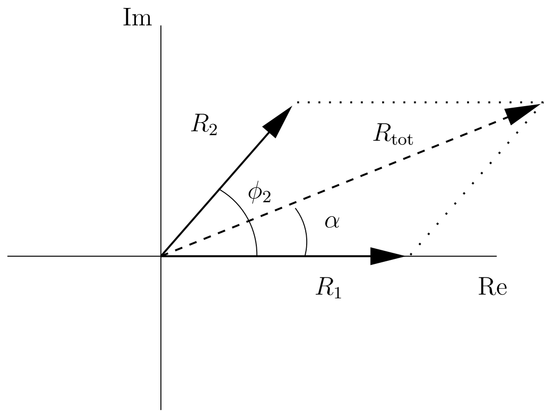


Figure 5.5: Phasors representing out of phase signals and their sum.

$\tilde{v}_2$ by an angle $\phi_2 - \alpha$). The angle α and the magnitude R_{tot} can be calculated using geometry and trigonometry in a manner that is analogous to the determination of the magnitude and direction when determining the sum of vectors.